

Symmetries and physical functions in general gauge theory

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Abstract

The aim of the present article is to describe the symmetry structure of a general gauge (singular) theory, and, in particular, to relate the structure of gauge transformations with the constraint structure of a theory in the Hamiltonian formulation. We demonstrate that the symmetry structure of a theory action can be completely revealed by solving the so-called symmetry equation. We develop a corresponding constructive procedure of solving the symmetry equation with the help of a special orthogonal basis for the constraints. Thus, we succeed in describing all the gauge transformations of a given action. We find the gauge charge as a decomposition in the orthogonal constraint basis. Thus, we establish a relation between the constraint structure of a theory and the structure of its gauge transformations. In particular, we demonstrate that, in the general case, the gauge charge cannot be constructed with the help of some complete set of first-class constraints alone, because the charge decomposition also contains second-class constraints. The above-mentioned procedure of solving the symmetry equation allows us to describe the structure of an arbitrary symmetry for a general singular action. Finally, using the revealed structure of an arbitrary gauge symmetry, we give a rigorous proof of the equivalence of two definitions of physicality condition in gauge theories: one of them states that physical functions are gauge-invariant on the extremals, and the other requires that physical functions commute with FCC (the Dirac conjecture).

1 Introduction

Most of the contemporary particle-physics theories are formulated as gauge theories. It is well-known that within the Hamiltonian formulation gauge theories are theories with constraints, in particular, first-class constraints (FCC). This fact is the main reason for a long and intensive study of the formal theory of constrained systems. The theory of constrained systems, initiated by the pioneering works of Bergmann and Dirac [1, 2] and then developed and presented in various review books [3, 4, 5, 6, 7], still attracts a great attention of researchers. Relatively simple were the first steps of the theory in formulating the dynamics of constrained systems in the phase space, thus elaborating the procedure of finding all the constraints (Dirac's procedure) and reorganizing the constraints to FCC and second-class constraints (SCC). From the very beginning, it became clear that the presence of FCC among the complete set of constraints in the Hamiltonian formulation is a direct indication that the theory is a gauge one, i.e., its Lagrangian action is invariant under gauge transformations, which, in the general case, are continuous transformations parameterized by arbitrary functions of time (of the space-time coordinates, in the case of field theory). It was demonstrated that the number of independent gauge parameters is equal to the number μ_1 of primary FCC, and the total number of nonphysical variables is equal to the number μ of all FCC, despite the fact that the equations of motion contain only μ_1 arbitrary functions of time (undetermined Lagrange multipliers to the primary FCC); see [6] and references therein. At the same time, we proved that for a class of theories for which the constraint structures of the complete theory and of its quadratic approximation are the same, and for which the constraint structure does not change from point to point in the phase-space (we call such theories perturbative ones), physical functions in the Hamiltonian formulation must commute with FCC. In a sense, this statement can be identified with the so-called Dirac conjecture. All models known until now in

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which the Dirac conjecture does not hold are nonperturbative in the above sense. After this preliminary progress in the theory of constrained systems, it became clear that a natural and very important continuation of studies is the attempt to relate the constraint structure and constraint dynamics of a gauge theory in the Hamiltonian formulation to specific features of a theory in the Lagrangian formulation, especially to relate the constraint structure to the structure of gauge transformations for a Lagrangian action. One of the key problems here is the following: how to construct a general expression for the gauge charge if the constraint structure in the Hamiltonian formulation is known? Another principle question closely related to the latter is: can one identify the physical functions defined as those commuting with FCC in the Hamiltonian formulation (the so-called physicality condition in the Hamiltonian sense, well-known as the Dirac conjecture) with the physical functions defined as gauge-invariant functions? (In what follows, we refer to the latter definition of physical functions as the physicality condition in the gauge sense.) Many efforts have been made in attempting to answer these questions: see, for example, [8]. All previous considerations contain some restricting assumptions about the structure of a theory (in particular, about the constraint structure), so that rigorous answers to all of the above questions are still unknown for a general gauge theory (even one belonging to the above-mentioned perturbative class). The aim of the present work is to answer the above questions for perturbative gauge theories in terms of rigorous statements.

In this connection, it should be noted that there exists an isomorphism between the symmetry classes of the Hamiltonian and Lagrangian actions of the same theory, since they are dynamically equivalent¹ (see [9]). It is then convenient to study the symmetry structure by considering the simpler Hamiltonian action (this is what we do in the present article), since it is a first-order action. The symmetries of the Lagrangian action can be obtained as a reduction of the Hamiltonian action symmetries by substituting all the Lagrange multipliers and momenta for coordinates and velocities. We demonstrate that the symmetry structure of the Hamiltonian action (and, hence, that of the Lagrangian action) can be completely revealed by solving the so-called symmetry equation. Choosing a special orthogonal basis for the constraints (introduced in [10] and described in sec. 3), one can analyze the symmetry equation algebraically. We develop the corresponding constructive procedure of solving the symmetry equation. Thus, we succeed in describing all the gauge transformations of a given action (sec. 4). We find the gauge charge as a decomposition in the orthogonal constraint basis. Thus, we establish the relation between the constraint structure of a theory and the structure of its gauge transformations. In particular, we demonstrate that, in the general case, the gauge charge cannot be constructed with the help of some complete set of FCC alone, since the decomposition also contains SCC. The above-mentioned procedure of solving the symmetry equation allows one to analyze the structure of any infinitesimal Noether symmetry. In doing this, we constructively demonstrate that any infinitesimal Noether symmetry can be represented as a sum of three kinds of symmetries: global, gauge, and trivial (sec. 5). In particular, we can see that the global part of a symmetry is a canonical transformation, which does not vanish on the extremals, and the corresponding conserved charge (the generator of this transformation) does not vanish on the extremals either. The gauge part of a symmetry does not vanish on the extremals, but the gauge charge does vanish on them. The trivial part of any symmetry vanishes on the extremals, and the corresponding charge vanishes quadratically on the extremals. In our procedure of solving the symmetry equation, the generators of canonical global and gauge symmetries may depend on Lagrange multipliers and their time derivatives. This happens in the case when the number of stages in the Dirac procedure is more than two. In addition, we prove that any infinitesimal Noether symmetry that vanishes on the extremals is a trivial symmetry (sec. 6). Finally, using the established structure of an arbitrary gauge symmetry, we strictly prove an equivalence of the two definitions of physicality condition in gauge theories: one of them states that physical functions are gauge-invariant on the extremals, and the other requires that physical functions commute with FCC (the Dirac conjecture) (sec. 7). In sec. 2, we present the basic notation and definitions.

¹Suppose that an action $S[q, y]$ contains two groups of coordinates q and y , such that the coordinates y can be expressed as local functions $y = \bar{y}(q^{[l]}, l < \infty)$ of q and their time derivatives with the help of the equations $\delta S / \delta y = 0$. We call y auxiliary coordinates. The action $S[q, y]$ and the reduced action $S[q] = S[q, \bar{y}]$ lead to the same equations for the coordinates q , see [14, ?]. The actions $S[q, y]$ and $S[q]$ are called dynamically equivalent actions.

2 Basic notation and definitions

We consider finite-dimensional systems described by a set of generalized coordinates $q \equiv \{q^a; a = 1, 2, \dots, n\}$. The following notation is used:

$$q^{a[l]} = (d_t)^l q^a, \quad l = 0, 1, \dots, \quad \left(q^{a[0]} = q^a \right), \quad d_t = \frac{d}{dt}. \quad (1)$$

We recall that the space of $q^{a[l]}$, $a = 1, \dots, n$, $l = 0, 1, \dots, N_a$, regarded as independent variables, is called the jet space [11]. The majority of physical quantities in classical mechanics are described by so-called local functions (LF) which are defined on the jet space. LF depend on $q^{a[l]}$, $l \leq N_a$, where $N_a < \infty$. We often denote LF as²

$$F(q^{a[0]}, q^{a[1]}, q^{a[2]}, \dots) = F(q^\square). \quad (2)$$

In our considerations, we use so-called local operators (LO). An LO \hat{U}_{Aa} is a matrix operator which acts on columns of LF f^a producing columns F_A of LF, $F_A = \hat{U}_{Aa} f^a$. Such LO have the form

$$\hat{U}_{Aa} = \sum_{k=0}^{K<\infty} u_{Aa}^k (d_t)^k, \quad (3)$$

where $u_{Aa}^k = u_{Aa}^k(q^\square)$ are LF. The operator

$$(\hat{U}^T)_{aA} = \sum_{k=0}^{K<\infty} (-d_t)^k u_{Aa}^k \quad (4)$$

is called the operator transposed to \hat{U}_{Aa} . The relation

$$F^A \hat{U}_{Aa} f^a = \left[(\hat{U}^T)_{aA} F^A \right] f^a + d_t Q, \quad (5)$$

where Q is an LF, holds for any LF F^A and f^a . The LO \hat{U}_{ab} is symmetric (+) or antisymmetric (-), respectively, if $(\hat{U}^T)_{ab} = \pm \hat{U}_{ab}$. Thus, for any antisymmetric LO \hat{U}_{ab} , relation (5) implies $f^a \hat{U}_{ab} f^b = dQ/dt$.

We say that $F_A(q^\square) = 0$ and $\chi_\alpha(q^\square) = 0$ are equivalent sets of equations (and denote this fact as $F = 0 \iff \chi = 0$) whenever they have the same sets of solutions. By $O(F)$ we denote any LF that vanishes on the equations $F_a(q^\square) = 0$. More exactly, we define $O(F) = \hat{V}^a F_a$, where \hat{V}^b are LO. Besides, we denote via $\hat{U} = \hat{O}(F)$ any LO that vanishes on the equations $F_a(q^\square) = 0$. That means that LF u that enter into the representation (3) for such an operator vanish on these equations, $u = O(F)$, or, equivalently $\hat{U}f = O(F)$ for any LF f .

We consider Lagrangian theories given by an Lagrangian action $S[q]$,

$$S[q] = \int_{t_1}^{t_2} L dt, \quad L = L(q^\square), \quad (6)$$

where the Lagrange function L is defined as an LF on the jet space. The Euler–Lagrange equations are

$$\frac{\delta S}{\delta q^a} = \sum_{l=0}^{\infty} (-d_t)^l \frac{\partial L}{\partial q^{a[l]}} = 0. \quad (7)$$

Any LF of the form $O(\delta S/\delta q)$ is called an extremal.

An infinitesimal transformation $q \rightarrow q' = q + \delta q$, with $\delta q = \delta q(q^\square)$, being an LF, is a symmetry of S in case

$$\delta L = d_t F, \quad (8)$$

where $F(q^\square)$ is an LF (such transformations are called Noether symmetries).

²Functions F may depend on time explicitly; however, we do not include t in the arguments.

Any infinitesimal symmetry implies a conservation law (Noether theorem):

$$\delta q^a \frac{\delta S}{\delta q^a} + d_t G = 0, \quad G = P - F. \quad (9)$$

Here, the LF F and P are

$$\delta L = d_t F, \quad P = \sum_a \sum_{m=1}^{N_a} p_a^m \delta q^{a[m-1]}, \quad p_a^m = \sum_{s=l}^{N_a} (-d_t)^{s-m} \frac{\partial L}{\partial q^{a[s]}}, \quad (10)$$

and the local function G (which is constant on the extremals) is referred to as a conserved charge related to the symmetry δq . In what follows, we call (9) the symmetry equation. The quantities δq , S , and G are related by the symmetry equation. Here and elsewhere, we often call the set δq , G a symmetry of the action S .

Noether infinitesimal symmetries can be global, gauge and trivial. Global symmetries have nonzero (on the extremals) conserved charges; they are parameterized by a set of time-independent parameters ν^α , $\alpha = 1, \dots, r$, and have the form $\delta q^a(t) = \rho_\alpha^a(t)\nu^\alpha$, where $\rho_\alpha^a(t)$ are generators of the global symmetry transformations. Gauge symmetries have zero (on the extremals) conserved charges; they are parameterized by time-dependent gauge parameters $\nu^\alpha(t)$, $\alpha = 1, \dots, r$, which are arbitrary functions of time (in the case of field theory, the gauge parameters depend on all space-time variables). Infinitesimal gauge transformations have the form

$$\delta q^a(t) = \hat{\mathfrak{R}}_\alpha^a(t)\nu^\alpha(t), \quad (11)$$

where the operators $\hat{\mathfrak{R}}_\alpha^a(t)$ are called generators of gauge transformations.

For any action, there exist trivial symmetry transformations $\delta_{\text{tr}}q$,

$$\delta_{\text{tr}}q^a = \hat{U}^{ab} \frac{\delta S}{\delta q^b}, \quad (12)$$

where \hat{U} is an antisymmetric LO, $(\hat{U}^T)^{ab} = -\hat{U}^{ab}$.

Trivial symmetries have zero (on the extremals) conserved charges; see below. Trivial symmetry transformations do not affect genuine trajectories. Two symmetry transformations $\delta_1 q$ and $\delta_2 q$ are called equivalent ($\delta_1 q \sim \delta_2 q$) whenever they differ by a trivial symmetry transformation: $\delta_1 q \sim \delta_2 q \iff \delta_1 q - \delta_2 q = \delta_{\text{tr}}q$. Thus, all the symmetry transformations of an action S can be divided into equivalence classes.

3 Symmetry equation and orthogonal constraint basis

We consider here the Hamiltonian formulation of a singular theory [6, 12]. The corresponding Hamiltonian action is denoted by S_H . We denote all irreducible constraints of the theory via $\Phi = (\Phi_l(\eta))$, where $\eta = (x, p)$ are all phase-space variables³,

$$\begin{aligned} \eta^A &= (x^a, p_a), \quad A = (\alpha, a), \quad \alpha = 1, 2, \quad a = 1, \dots, n, \\ \text{such that : } \eta^{1a} &= x^a, \quad \eta^{2a} = p_a. \end{aligned} \quad (13)$$

We suppose that the constraints of the theory are reorganized so that they can be divided into FCC, $\chi(\eta)$, and SCC, $\varphi(\eta)$. Thus, $\Phi = (\chi, \varphi)$. The Hamiltonian action and the corresponding equations of motion are

$$\begin{aligned} S_H[\boldsymbol{\eta}] &= \int \left[p\dot{x} - H^{(1)}(\boldsymbol{\eta}) \right] dt, \quad \boldsymbol{\eta} = (\eta, \lambda), \\ H^{(1)}(\boldsymbol{\eta}) &= H(\eta) + \lambda\Phi^{(1)}(\eta); \\ \frac{\delta S_H}{\delta \boldsymbol{\eta}} = 0 &\implies \begin{cases} \dot{\eta} = \{\eta, H^{(1)}\} \\ \Phi^{(1)}(\eta) = 0 \end{cases}, \end{aligned} \quad (14)$$

³In the general case of theories with higher derivatives, the space of x is larger than the space of generalized coordinates q , see, e.g. [12].

where $H^{(1)}$ is the total Hamiltonian, $\Phi^{(1)}$ are primary constraints, and λ are Lagrange multipliers to the primary constraints.

In the general case, the Hamiltonian H and the constraints Φ can depend on time t explicitly. We take such a possibility into account. However, the argument t will not be written explicitly in what follows. Using Eq. (14), we can write the total time derivative of a function $f(\eta)$ as

$$d_t f = \partial_t f + \dot{\eta} \partial_\eta f = \left\{ f, H^{(1)} + \epsilon \right\} + (\dot{\eta} - \{\eta, H^{(1)}\}) \partial_\eta f.$$

Here, ϵ is the (formal) momentum conjugate to time t , and the Poisson brackets are defined in the extended phase space of the variables $\eta; t; \epsilon$; for details, see [6].

If $\delta\eta = (\delta x, \delta p, \delta\lambda)$ is a symmetry of the Hamiltonian action S_H , then the symmetry equation is

$$\delta\eta \frac{\delta S_H}{\delta\eta} + d_t G = 0, \quad (15)$$

where G is the corresponding conserved charge. One can study the symmetry of the action S_H by solving this symmetry equation. If $\delta\eta$ is a symmetry of the Hamiltonian action S_H , then symmetries $\delta q_L(q^\square)$ of the Lagrangian action S can be obtained by writing the Hamiltonian symmetries $\delta q(\eta^\square)$, $\delta q \in \delta x$, as functions on the jet space q^\square , see [9].

For further consideration, especially for solving the symmetry equation, it is convenient to accept that the set of all the constraints Φ is already reorganized to the special form consistent with the Dirac procedure; see [10]. We call such a reorganized set of constraints an orthogonal constraint basis. In this case,

$$\Phi = \left(\Phi^{(i)} \right), \quad \Phi^{(i)} = (\varphi^{(i)}; \chi^{(i)}), \quad i = 1, \dots, \aleph. \quad (16)$$

Such $\chi^{(i)}$ and $\varphi^{(i)}$ are, respectively, the FCC and SCC of the i -th stage of the Dirac procedure, while \aleph is the number of the final stage. The total Hamiltonian is

$$H^{(1)} = H + \lambda \Phi^{(1)} = H + \lambda_\varphi \varphi^{(1)} + \lambda_\chi \chi^{(1)}, \quad (17)$$

where $\chi^{(1)}$ are the primary FCC, and $\varphi^{(1)}$ are the primary SCC; $\lambda = (\lambda_\varphi, \lambda_\chi)$, while λ_φ and λ_χ are the corresponding Lagrange multipliers to the primary SCC and FCC, respectively. At each stage of the Dirac procedure, the constraints are divided into groups:

$$\begin{aligned} \varphi^{(i)} &= \left(\varphi^{(i|s)} \right), \quad s = i, \dots, \aleph_\varphi; \\ \chi^{(i)} &= \left(\chi^{(i|a)} \right), \quad a = i, \dots, \aleph_\chi, \end{aligned} \quad (18)$$

where \aleph_φ and \aleph_χ stand for the numbers of the final stages at which SCC and new FCC, respectively, still appear. We can write⁴:

$$[\varphi^{(i)}] = 0, \quad i > \aleph_\varphi; \quad [\chi^{(i)}] = 0, \quad i > \aleph_\chi, \quad \aleph = \max(\aleph_\varphi, \aleph_\chi).$$

In what follows, we often use the notation

$$\lambda_{\varphi^{(1|s)}} = \lambda_s, \quad \lambda_{\chi^{(1|a)}} = \lambda^a,$$

so that

$$H^{(1)} = H + \lambda_s \varphi^{(1|s)} + \lambda^a \chi^{(1|a)}. \quad (19)$$

The division (18) produces chains of constraints. All constraints in a chain are of the same class. One ought to say that the numbers of constraints at each stage in the same chain are equal. At the same time, each chain may either be empty or contain several functions. Therefore, whenever there exist FCC (SCC), the corresponding primary FCC (SCC) exist as well.

⁴The following notation is used: suppose that $F_a(\eta)$, $a = 1, \dots, n$, are some functions, then $[F]$ is the number of these functions, $[F] = n$. Note that the brackets \square are also used to denote time derivatives ($q^{[l]} = (d_t)^l q$) and the arguments of an action functional (e.g. $S[q]$).

There exist \aleph_φ chains of SCC,

$$\varphi^{(\dots|s)} = \left(\varphi^{(i|s)}, i = 1, \dots, s \right), s = 1, \dots, \aleph_\varphi,$$

labeled by the index s , and \aleph_χ chains of FCC,

$$\chi^{(\dots|a)} = \left(\chi^{(i|a)}, i = 1, \dots, a \right), a = 1, \dots, \aleph_\chi,$$

labeled by the index a . The length of the longest chain of SCC is \aleph_φ , and the length of the longest chain of FCC is \aleph_χ . The orthogonal constraint basis has the following properties:

$$\begin{aligned} & \left\{ \begin{aligned} & \left\{ \varphi^{(i|s)}, \bar{H} + \epsilon \right\} = \varphi^{(i+1|s)} + O(\Phi^{(\dots|i)}) \\ & \left\{ \varphi^{(i|s)}, \Phi^{(1)} \right\} = O(\Phi^{(\dots|i)}) \end{aligned} \right\}, i = 1, \dots, \aleph_\varphi - 1, s = i + 1, \dots, \aleph_\varphi, \\ & \left\{ \begin{aligned} & \left\{ \chi^{(i|a)}, \bar{H} + \epsilon \right\} = \chi^{(i+1|a)} + O(\Phi^{(\dots|i)}) \\ & \left\{ \chi^{(i|a)}, \Phi^{(1)} \right\} = O(\Phi^{(\dots|i)}) \end{aligned} \right\}, i = 1, \dots, \aleph_\chi - 1, a = i + 1, \dots, \aleph_\chi, \\ & \left\{ \begin{aligned} & \left\{ \chi^{(i|i)}, \bar{H} + \epsilon \right\} = O(\Phi^{(\dots|i)}) \\ & \left\{ \chi^{(i|i)}, \Phi^{(1)} \right\} = O(\Phi^{(\dots|i)}) \end{aligned} \right\}, i = 1, \dots, \aleph_\chi, \\ & \left\{ \begin{aligned} & \left\{ \varphi^{(i|s)}, \varphi^{(j|v)} \right\} = \delta_{s,v} \delta_{i,s+1-j} \theta^{(s)} + O(\Phi) \\ & i \leq s, j \leq v, s, v = 1, \dots, \aleph_\varphi. \end{aligned} \right\}. \end{aligned} \quad (20)$$

Here, $\Phi^{(\dots|i)} = (\Phi^{(1)}, \dots, \Phi^{(i)})$,

$$\bar{H} = H + \bar{\lambda}_s \varphi^{(1|s)}, \quad (21)$$

where $\bar{\lambda}_s = \bar{\lambda}_s(\eta)$ are expressions for the Lagrange multipliers to the primary SCC that are determined by the Dirac procedure, and $\theta^{(s)}$ is a nonsingular matrix. It should be noted that the Hamiltonian \bar{H} differs from the total Hamiltonian $H^{(1)}$ by terms quadratic in the extremals and by FCC:

$$H^{(1)} = \bar{H} + \Lambda_s \varphi^{(1|s)} + \lambda^a \chi^{(1|a)}, \Lambda_s = \lambda_s - \bar{\lambda}_s = O\left(\frac{\delta S_H}{\delta \eta}\right). \quad (22)$$

From the properties of the orthogonal constraint basis, one can see that the Poisson brackets of SCC from different chains vanish on the constraint surface. Within the Dirac procedure, the group $\varphi^{(1|s)}$ of primary SCC produces SCC of the second stage, third stage, and so on, which belong to the same chain, $\varphi^{(1|s)} \rightarrow \varphi^{(2|s)} \rightarrow \varphi^{(3|s)} \rightarrow \dots \rightarrow \varphi^{(s|s)}$. The chain of SCC labeled by the number s ends with the group of the s -th-stage constraints. The consistency conditions for the SCC $\varphi^{(i|i)}$ of the i -th stage determine the Lagrange multipliers λ_i to be $\bar{\lambda}_i$. We stress that the consistency conditions for the SCC $\varphi^{(i|s)}$, $s > i$, of the i -th stage produce SCC $\varphi^{(i+1|s)}$ of the $i+1$ -th stage.

At the same time, the group $\chi^{(1|a)}$ of primary FCC produces FCC of the second stage, third stage, and so on, which belong to the same chain, $\chi^{(1|a)} \rightarrow \chi^{(2|a)} \rightarrow \chi^{(3|a)} \rightarrow \dots \rightarrow \chi^{(a|a)}$. The consistency conditions for the FCC $\chi^{(i|s)}$, $s > i$, of the i -th stage produce the FCC $\chi^{(i+1|s)}$ of the $i+1$ -th stage. The chain of FCC labeled by the number a ends with the group of the a -th-stage constraints. The consistency conditions for the FCC $\chi^{(i|i)}$ of the i -th stage do not produce any new constraints and do not determine any Lagrange multipliers. Thus, the Lagrange multipliers λ_χ are not determined by the Dirac procedure (or by the complete set of equations of motion).

The described hierarchy of constraints in the orthogonal basis (within the Dirac procedure) looks schematically as follows:

$$\begin{array}{ccccccc} \varphi^{(1|1)} & \rightarrow & \bar{\lambda}_1 & & & & \\ \varphi^{(1|2)} & \rightarrow & \varphi^{(2|2)} & \rightarrow & \bar{\lambda}_2 & & \\ \vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots \\ \varphi^{(1|\aleph-1)} & \rightarrow & \varphi^{(2|\aleph-1)} & \rightarrow & \varphi^{(3|\aleph-1)} & \dots & \varphi^{(\aleph-1|\aleph-1)} \rightarrow \bar{\lambda}_{\aleph-1} \\ \varphi^{(1|\aleph)} & \rightarrow & \varphi^{(2|\aleph)} & \rightarrow & \varphi^{(3|\aleph)} & \dots & \varphi^{(\aleph-1|\aleph)} \rightarrow \varphi^{(\aleph|\aleph)} \rightarrow \bar{\lambda}_\aleph \\ \chi^{(1|\aleph)} & \rightarrow & \chi^{(2|\aleph)} & \rightarrow & \chi^{(3|\aleph)} & \dots & \chi^{(\aleph-1|\aleph)} \rightarrow \chi^{(\aleph|\aleph)} \rightarrow O(\Phi^{(\dots|\aleph)}) \\ \chi^{(1|\aleph-1)} & \rightarrow & \chi^{(2|\aleph-1)} & \rightarrow & \chi^{(3|\aleph-1)} & \dots & \chi^{(\aleph-1|\aleph-1)} \rightarrow O(\Phi^{(\dots|\aleph-1)}) \\ \vdots & \rightarrow & \vdots & \rightarrow & \vdots & \rightarrow & \vdots \\ \chi^{(1|2)} & \rightarrow & \chi^{(2|2)} & \rightarrow & O(\Phi^{(\dots|2)}) & & \\ \chi^{(1|1)} & \rightarrow & O(\Phi^{(1)}) & & & & \end{array}$$

We note that the properties of the orthogonal constraint basis are extremely helpful in analyzing the symmetry equation; in particular, this equation in the orthogonal basis can be solved algebraically (see below). The properties of the basis allow one to conjecture (and then strictly prove) the form of the conserved charges as decompositions in the orthogonal constraint basis. For example, these properties imply that the SCC $\varphi^{(i|i)}$ cannot enter linearly into the conserved charges. At the same time, one can see that only the FCC $\chi^{(i|i)}$ enter the gauge charges multiplied by independent gauge parameters; other FCC $\chi^{(i|a)}$, $a > i$, are multiplied by factors that must contain derivatives of the same gauge parameters.

It is also important to recall that there exists a canonical transformation from the initial phase-space variables η to the special phase-space variables $\vartheta = (\omega, Q, \Omega)$; see [6]. The variables ω are physical, and the variables Ω define the constraint surface by the equations $\Omega = 0$. The dynamics of the physical variables ω is governed by the physical action $S_{\text{Ph}}[\omega]$,

$$S_{\text{Ph}}[\omega] = S_{\text{H}}|_{\Omega=0}. \quad (23)$$

The variables Q are nonphysical. In theories with FCC, the actions S and S_{H} are dynamically equivalent, while are both dynamically nonequivalent to the physical action S_{Ph} .

In what follows, we use a set of LF, Γ ,

$$\Gamma = (\Phi, J) = O\left(\frac{\delta S_{\text{H}}}{\delta \eta}\right), \quad J = (\mathcal{I}, \Lambda), \quad \mathcal{I} = \dot{\eta} - \{\eta, \bar{H}\} - \left\{\eta, \chi^{(1|a)}\right\} \lambda^a, \quad (24)$$

for the complete set of extremals. Taking (14) into account, one can easily verify that the set Γ is equivalent to the complete set of extremals $\delta S_{\text{H}}/\delta \eta$. We also note that the set of variables η^{\square} , λ_s^{\square} , $\lambda^a{}^{\square}$ is equivalent to the set η , J^{\square} , $\lambda^a{}^{\square}$. Further, we often use the latter variables to analyze the symmetry equation.

4 Gauge symmetries

Analyzing the symmetry equation, we prove below the following assertion.

In theories with FCC, there exist nontrivial symmetries $\delta_\nu \eta$, G_ν of the Hamiltonian action S_{H} that are gauge transformations. These symmetries are parameterized by gauge parameters ν . These parameters are arbitrary functions of time⁵ and arbitrary LF of η^{\square} ,

$$\nu = \nu_{(a)\sigma_a}(t, \eta^{\square}), \quad a = 1, \dots, \aleph_{\chi}.$$

The gauge parameters are labeled by an index a (the number of the corresponding FCC chain) and by a fine index⁶ σ_a that labels FCC in the chain a . The complete number of all gauge parameters is equal to the number of all primary FCC:

$$[\nu] = [\chi^{(1)}]. \quad (25)$$

The corresponding conserved charge (gauge charge) is an LF, $G_\nu = G_\nu(\eta, \lambda^{\square}, \nu^{\square})$, which vanishes on the extremals. The gauge charge has the following representation in terms of the orthogonal constraint basis:

$$G_\nu = \sum_{a=1}^{\aleph_{\chi}} \nu_{(a)} \chi^{(a|a)} + \sum_{i=1}^{\aleph_{\chi}-1} \sum_{a=i+1}^{\aleph_{\chi}} C_{i|a}^\chi \chi^{(i|a)} + \sum_{i=1}^{\aleph_{\chi}-1} \sum_{s=i+1}^{\aleph_{\chi}} C_{i|s}^\varphi \varphi^{(i|s)}. \quad (26)$$

Here, $C_{i|s}^\varphi = C_{i|s}^\varphi(\eta, \lambda^{\square}, \nu^{\square})$ and $C_{i|a}^\chi = C_{i|a}^\chi(\eta, \lambda^{\square}, \nu^{\square})$ are some LF, which can be determined from the symmetry equation in an algebraic way. The gauge charge depends both on the gauge parameters and on their time derivatives up to the order $\aleph_{\chi} - 1$,

$$G_\nu = \sum_{a=1}^{\aleph_{\chi}} \sum_{m=0}^{a-1} G_m^a(\eta, \lambda^{\square}) \nu_{(a)}^{[m]}. \quad (27)$$

⁵We have included t in the arguments of the functions ν explicitly, in contrast to other cases, to emphasize this dependence.

⁶Sometimes, the fine index σ_i is omitted, but summation over the index i always assumes the summation over σ_i as well.

Here, $G_m^a(\eta, \lambda^\square)$ are some LF. The total number of independent gauge parameters, together with their time derivatives that essentially enter in the gauge charge, is equal to the number of all FCC:

$$\sum_{m=0}^{n_\chi} [\nu^{[m]}] = [\chi]. \quad (28)$$

The gauge charge is the generating function for the variations $\delta\eta$ of the phase-space variables:

$$\delta_\nu \eta = \{\eta, G_\nu\} = \{\eta, \eta^A\} \frac{\partial G_\nu}{\partial \eta^A}. \quad (29)$$

(Note that the Poisson bracket in (29) acts only on the explicit dependence of the gauge charge of η .) Therefore, the total number of independent gauge parameters, together with their time derivatives that essentially enter in the variations $\delta_\nu \eta$, is also equal to the number of all FCC. The variations $\delta_\nu \lambda$ contain an additional time derivative of the gauge parameters, namely, they have the form

$$\delta_\nu \lambda = \sum_{a=1}^{n_\chi} \sum_{m=0}^a \Upsilon_m^a \nu_{(a)}^{[m]}, \quad (30)$$

where $\Upsilon_m^a = \Upsilon_m^a(\eta, \lambda^\square)$ are some LF.

The above assertion is proved below. At the same time, we present a constructive procedure of finding the gauge transformations, based on solving the symmetry equation in terms of the orthogonal constraint basis.

Consider the symmetry equation (15). Taking into account the action structure (14) and the anticipated form (29) of the variations $\delta\eta$, we rewrite this equation as follows:

$$\hat{H}G - \lambda^a \left\{ \chi^{(1|a)}, G \right\} - \Lambda_s \{ \varphi^{(1|s)}, G \} = \varphi^{(1|s)} \hat{\delta}\Lambda_s + \chi^{(1|a)} \delta\lambda^a, \quad \hat{\delta}\Lambda_s = \delta\lambda_s - \{ \bar{\lambda}_s, G \}, \quad (31)$$

where the operator \hat{H} is defined on any LF $F(\eta, \lambda^\square, \nu^\square)$ as

$$\hat{H}F = \{ F, \bar{H} + \epsilon \} + \frac{\partial F}{\partial \lambda^{[m]}} \lambda^{[m+1]} + \frac{\partial F}{\partial \nu^{[m]}} \nu^{[m+1]}. \quad (32)$$

Let us try to solve equation (31) with the gauge charge G_ν of the form (26). Thus, we obtain the following equation for the LF $\mathbf{C}_i = (C_{i|s}^\varphi, C_{i|a}^\chi)$:

$$\begin{aligned} & \sum_{i=1}^{n_\chi-1} \left(\mathbf{C}_i \left[\Phi^{(i+1)} + O(\Phi^{(\dots i)}) \right] + \Phi^{(i)} \left[\hat{H} \mathbf{C}_i - \lambda^a \left\{ \chi^{(1|a)}, \mathbf{C}_i \right\} - \Lambda_s \{ \varphi^{(1|s)}, \mathbf{C}_i \} \right] \right) \\ & + \Pi_{n_\chi} = \varphi^{(1|s)} \hat{\delta}\Lambda_s + \chi^{(1|a)} \delta\lambda^a, \end{aligned} \quad (33)$$

where

$$\Pi_{n_\chi} = \sum_{a=1}^{n_\chi} \left[Z^a + \chi^{(a|a)} d_t \right] \nu_{(a)}, \quad Z^a = \hat{H} \chi^{(a|a)} - \lambda^b \left\{ \chi^{(1|b)}, \chi^{(a|a)} \right\} - \Lambda_s \{ \varphi^{(1|s)}, \chi^{(a|a)} \} = O(\Phi^{(\dots a)}).$$

Considering equation (33) on the constraint surface $\Phi^{(\dots n_\chi-1)} = 0$, we obtain

$$\mathbf{C}_{n_\chi-1} \Phi^{(n_\chi)} + \nu_{(n_\chi)} \kappa \Phi^{(n_\chi)} + \nu_{(n_\chi)}^{[1]} \chi^{(n_\chi|n_\chi)} = 0$$

with allowance made for the relation

$$\mathbf{Z}^{n_\chi} \Big|_{\Phi^{(\dots n_\chi-1)}=0} = O(\Phi^{(n_\chi)}) = \kappa \Phi^{(n_\chi)}. \quad (34)$$

where κ is a matrix with elements that are LF. Therefore, we can determine $\mathbf{C}_{n_\chi-1}$ as linear combinations of $\nu_{(n_\chi)}$ and $\nu_{(n_\chi)}^{[1]}$ to obey the latter equation. Substituting thus determined $\mathbf{C}_{n_\chi-1}$ into (33), we arrive at the equation

$$\begin{aligned} & \sum_{i=1}^{n_\chi-2} \left[\mathbf{C}_i \left[\Phi^{(i+1)} + O(\Phi^{(\dots i)}) \right] + \Phi^{(i)} \left[\hat{H} \mathbf{C}_i - \lambda^a \left\{ \chi^{(1|a)}, \mathbf{C}_i \right\} - \Lambda_s \{ \varphi^{(1|s)}, \mathbf{C}_i \} \right] \right] \\ & + \Pi_{n_\chi-1} = \varphi^{(1|s)} \hat{\delta}\Lambda_s + \chi^{(1|a)} \delta\lambda^a, \end{aligned} \quad (35)$$

where

$$\Pi_{N_x-1} = \sum_{a=1}^{N_x-1} \left[Z^a + \chi^{(a|a)} d_t \right] \nu_{(a)} + \sum_{k=0}^2 \phi_k \nu_{(N_x)}^{[k]}, \quad \phi_k = O\left(\Phi^{(\dots N_x-1)}\right).$$

Considering equation (35) on the constraint surface $\Phi^{(\dots N_x-2)} = 0$, we obtain

$$\begin{aligned} & C_{N_x-2} \Phi^{(N_x-1)} + \nu_{(N_x-1)}^{[1]} \chi^{(N_x-1|N_x-1)} \\ & + \left. \left(\nu_{(N_x-1)} Z^{N_x-1} + \sum_{k=0}^2 \phi_k \nu_{(N_x)}^{[k]} \right) \right|_{\Phi^{(\dots N_x-2)}=0} = 0. \end{aligned} \quad (36)$$

By analogy with (34), we find that all matrix LF Z^{N_x-1} and ϕ_k are proportional to the constraints $\Phi^{(N_x-1)}$. Therefore, we can determine C_{N_x-2} as linear combinations of $\nu_{(N_x-1)}$, $\nu_{(N_x-1)}^{[1]}$, and $\nu_{(N_x)}^{[1]}$, $\nu_{(N_x)}^{[2]}$ to obey Eq. (36). Proceeding in the same manner, we determine any C_i as the following linear combinations of the gauge parameters and their time derivatives:

$$C_i = \sum_{a=i+1}^{N_x} \sum_{m=0}^{a-i} S_{im}^a \nu_{(a)}^{[m]}. \quad (37)$$

Here, $S_{im}^a = S_{im}^a(\eta, \lambda^{\parallel})$ are some LF.

In addition, we can see that

$$C^\varphi = O(\Gamma). \quad (38)$$

Indeed, Eq. (38) follows from the relations (we recall that symmetry variations of extremals are proportional to extremals)

$$\hat{\delta}\varphi^{(s+1-i|s)} = O(\Gamma) = \left\{ \varphi^{(s+1-i|s)}, G_\nu \right\} = C_{i|s}^\varphi \left\{ \varphi^{(s+1-i|s)}, \varphi^{(i|s)} \right\} + O(\Phi),$$

with allowance made for (20).

Finally, we conclude that the gauge charge G_ν has the following representation:

$$G_\nu = \sum_{m=1}^{N_x} \sum_{a=m}^{N_x} G^{ma} \nu_{(a)}^{[m-1]}. \quad (39)$$

Here, $G^{ma} = G^{ma}(\eta, \lambda^{\parallel})$ are LF of the form

$$G^{ma} = \sum_{k=1}^{N_x} \sum_{b=k}^{N_x} \chi^{(k|b)} \mathcal{C}_{kb}^{ma} + O(\Gamma^2) = O(\chi) + O(\Gamma^2), \quad (40)$$

where $\mathcal{C}_{kb}^{ma} = \mathcal{C}_{kb}^{ma}(\eta, \lambda^{\parallel})$ are some LF. Then the form of the variations $\delta_\nu \eta$ follows from (29),

$$\delta_\nu \eta = \left(\sum_{k=1}^{N_x} \sum_{b=k}^{N_x} \right) \left(\sum_{m=1}^{N_x} \sum_{a=m}^{N_x} \right) \left\{ \eta, \chi^{(k|b)} \right\} \mathcal{C}_{kb}^{ma} \nu_{(a)}^{[m-1]} + O(\Gamma). \quad (41)$$

After substituting the obtained functions C_i back into equation (33), its left-hand side turns out to be proportional to primary constraints. Since these constraints are linearly independent by construction, we can find all the variations $\delta_\nu \lambda$ from this equation. Their general structure is given by Eq. (30). In particular, one can see that

$$\delta_\nu \lambda^a = \sum_{b=1}^{N_x} \mathcal{D}^{ab} \nu_{(b)}^{[b]} + O(\nu_{(j)}^{[l]}), \quad l < j, \quad (42)$$

where $\mathcal{D}^{ab} = \mathcal{D}^{ab}(\eta, \lambda^{\parallel})$ are some LF. Note that the LF G^{ma} and Υ_m^a (as well as \mathcal{C}_{kb}^{ma} and \mathcal{D}^{ab}) do not depend on the gauge parameters and are, in this sense, universal.

The matrices \mathcal{C} and \mathcal{D} are nonsingular. Indeed, as was demonstrated in [13], they are constant matrices of the form

$$\mathcal{C}_{ka}^{mb} = \delta_{b,a}\delta_{m,b-k+1}, \quad \mathcal{D}^{ab} = \delta_{a,b}$$

in the quadratic approximation. Since these matrices are nonsingular in the quadratic approximation, they are nonsingular in the exact perturbative theory at least in a vicinity of the consideration point, which can be chosen as the zero point of the jet space of η^{\square} .

The gauge charge and the variations $\delta_\nu\eta$ essentially depend on all gauge parameters and their time derivatives $\nu_{(a)}^{[m]}$, $m = 0, \dots, a-1$, $a = 1, \dots, \aleph_\chi$, while the variations $\delta_\nu\lambda^a$ essentially depend on all derivatives $\nu_{(a)}^{[a]}$. Indeed,

$$\begin{aligned} \frac{\partial G_\nu}{\partial \nu_{(a)}^{[m]}} &= \mathcal{P}^{(a-m|a)} + O(\eta^2), \quad m = 0, \dots, a-1, \\ \frac{\partial (\delta_\nu\lambda^a)}{\partial \nu_{(b)}^{[b]}} &= \delta_{a,b} + O(\eta), \quad a, b = 1, \dots, \aleph_\chi. \end{aligned}$$

As one can see from the structure of the gauge charge (in the special phase space variables $\vartheta = (\omega, Q, \Omega)$; see [6]), the gauge transformations, taken on the extremals, transform only the nonphysical variables Q and λ^a .

Below, we are going to study the structure of arbitrary symmetry of the general Hamiltonian action S_H . In particular, we prove that, with accuracy up to a trivial transformation, any gauge transformation can be represented in the form (29), (30), with the gauge charge (26).

5 Structure of arbitrary symmetry

We prove below the following assertion.

Any symmetry $\delta\eta$, G of the Hamiltonian action S_H can be represented as the sum of three types of symmetries,

$$\begin{pmatrix} \delta\eta \\ G \end{pmatrix} = \begin{pmatrix} \delta_c\eta \\ G_c \end{pmatrix} + \begin{pmatrix} \delta_{\bar{\nu}}\eta \\ G_{\bar{\nu}} \end{pmatrix} + \begin{pmatrix} \delta_{\text{tr}}\eta \\ G_{\text{tr}} \end{pmatrix}, \quad (43)$$

such that:

The set $\delta_c\eta$, G_c is a global symmetry, canonical for the phase-space variables η . The corresponding conserved charge G_c does not vanish on the extremals.

The set $\delta_{\bar{\nu}}\eta$, $G_{\bar{\nu}}$ is a gauge transformation presented in the previous section, with fixed gauge parameters (i.e., with a specific form of the functions $\nu = \bar{\nu}(t, \eta^{\square})$) that do not vanish on the extremals (in what follows, we call such a symmetry a particular gauge transformation). The corresponding conserved charge $G_{\bar{\nu}}$ vanishes on the extremals, whereas the variations $\delta_{\bar{\nu}}\eta$ do not.

The set $\delta_{\text{tr}}\eta$, G_{tr} is a trivial symmetry. All the variations $\delta_{\text{tr}}\eta$ and the corresponding conserved charge G_{tr} vanish on the extremals. The gauge charge G_{tr} depends on the extremals as $G_{\text{tr}} = O(\Gamma^2)$.

Below, we prove the above assertions and present a constructive way of finding the components of the decomposition (43). The procedure can be divided into the four steps:

5.1 Constructing the function $G_J(\eta)$

Supposing that $\delta\eta$, G is a symmetry, and taking into account the structure of the total Hamiltonian (22), we can write the symmetry equation (15) as

$$\delta\eta E^{-1}\mathcal{I} - \varphi^{(1|s|)}\hat{\delta}\Lambda_s - \Lambda_s\hat{\delta}\varphi^{(1|s|)} - \chi^{(1|a|)}\delta\lambda^a + d_t G = 0, \quad \hat{\delta}\Lambda_s = \delta\lambda_s - \frac{\partial\bar{\lambda}_s}{\partial\eta}\delta\eta. \quad (44)$$

In our consideration, we use the set of variables η , J^{\square} , $\lambda^a{}^{\square}$, see sec. 3. We denote via $\delta_J\eta$, G'_J the corresponding zero-order terms in the decomposition of the quantities $\delta\eta$, G with respect to the extremals J^{\square} ,

$$\begin{pmatrix} \delta\eta \\ G \end{pmatrix} = \begin{pmatrix} \delta_J\eta + O(J) \\ G'_J + B_m J^{[m]} + O(J^2) \end{pmatrix}, \quad (45)$$

where $\delta_J \eta = \delta_J \eta(\eta, \lambda^a)$, $G'_J = G'_J(\eta, \lambda^a)$, and $B_m = B_m(\eta, \lambda^a)$. We then rewrite equation (44), retaining only the terms of zero and first order with respect to the extremals J^a . Such an equation has the form

$$\begin{aligned} \delta_J \eta E^{-1} \mathcal{I} - \chi^{(1|a)} \delta_J \lambda^a &= -\hat{H} G'_J + \left\{ \chi^{(1|a)}, G'_J \right\} \lambda^a + \Lambda_s \{ \varphi^{(1|s)}, G'_J \} \\ &+ \{ \eta, G'_J \} E^{-1} \mathcal{I} - J^{[m]} \hat{H} B_m + \lambda^a \left\{ \chi^{(1|a)}, B_m \right\} J^{[m]} - B_m J^{[m+1]} + O(\Phi \Gamma). \end{aligned} \quad (46)$$

Contributions from the terms $\varphi^{(1|s)} \hat{\delta} \Lambda_s$ and $\Lambda_s \hat{\delta} \varphi^{(1|s)}$ are accumulated in $O(\Phi \Gamma)$, and the operator \hat{H} is defined by (32).

Analyzing the terms with the extremals J^a (starting from the highest derivative) in Eq. (46), we can see that $B_m = O(\Phi)$ for every m . Considering then the terms proportional to \mathcal{I} in Eq. (46), we get the following expression:

$$\delta_J \eta = \{ \eta, G'_J \} + O(\Phi)$$

for the variations $\delta_J \eta$. We then see that

$$\{ \Phi, G'_J \} = O(\Phi) \quad (47)$$

with allowance made for the relations

$$\hat{\delta} \Phi = O(\Gamma) \implies \hat{\delta}_J \Phi = O(\Phi) = \{ \Phi, G'_J \} + O(\Phi).$$

We can check that $\{ \chi, G'_J \}$ are first-class functions, which implies that

$$\{ \chi, G'_J \} = O(\chi) + O(\Phi^2). \quad (48)$$

Considering the remaining terms in Eq. (46), we get the equation

$$\chi^{(1|a)} \delta_J \lambda^a = \hat{H} G'_J + \lambda^a \left\{ \chi^{(1|a)}, G'_J \right\} + O(\Phi^2), \quad (49)$$

which relates $\delta_J \lambda^a$ and G'_J . This equation allows us to study the function G'_J in more detail. To this end, we rewrite the equation as

$$\{ G'_J, \bar{H} + \epsilon \} + \frac{\partial G'_J}{\partial \lambda^a} \lambda^{a[m+1]} = O(\chi) + O(\Phi^2),$$

taking into account (32) and (48). Analyzing terms with the Lagrange multipliers λ_χ^a (starting from the highest derivatives) in this equation, we can see that these multipliers can enter only the terms that vanish on the constraint surface. For example, considering the terms with the highest derivative $\lambda_\chi^{[M+1]}$ in the latter equation, we obtain

$$\frac{\partial G'_J}{\partial \lambda^a} = O(\chi) + O(\Phi^2) \implies G'_J = G'_J(\cdots \lambda^{a[M-1]}) + O(\chi) + O(\Phi^2).$$

Similarly, we finally conclude that G'_J has the structure

$$G'_J = G_J(\eta) + O(\chi) + O(\Phi^2) = G_J(\eta) + B_\chi \chi + O(\Phi^2), \quad (50)$$

where $B_\chi = B_\chi(\eta, \lambda^a)$. For the function G_J , we obtain

$$\{ \varphi, G_J \} = O(\Phi), \quad \{ \chi, G_J \} = O(\chi) + O(\Phi^2) \quad (51)$$

with allowance made for (47) and (48).

The relation

$$\delta \Lambda = O(\Gamma) \implies \delta_J \lambda_s - \{ \bar{\lambda}_s, G_J + B_\chi \chi \} = O(\Gamma) = O(\Phi),$$

defines the variations $\delta_J \lambda_s$, taken in the lowest order with respect to the extremals.

In addition to relations (51), the function $G_J(\eta)$ also obeys

$$\{ G_J, \bar{H} + \epsilon \} = O(\chi) + O(\Phi^2) \quad (52)$$

Thus, the above consideration allows one to represent a refined version of the representation (45)

$$\begin{pmatrix} \delta\eta \\ \delta\lambda_s \\ \delta\lambda^a \\ G \end{pmatrix} = \begin{pmatrix} \{\eta, G_J + B_\chi\chi\} + O(\Gamma) \\ \{\bar{\lambda}_s, G_J + B_\chi\chi\} + O(\Gamma) \\ \delta_J\lambda^a + O(J) \\ G_J + B_\chi\chi + O(\Gamma^2) \end{pmatrix}, \quad (53)$$

where $\delta_J\lambda^a = \delta_J\lambda^a(\eta, \lambda^b)$.

5.2 Constructing the function $G_\Gamma(\eta)$

We now select from the function G_J a part G_Γ that does not vanish on the constraint surface:

$$G_J(\eta) = G_\Gamma(\eta) + G_1(\eta), \quad G_\Gamma = G_J|_{\Phi=0}. \quad (54)$$

The function G_1 vanishes on this surface. Certainly, this decomposition is not unique. We fix the procedure of such a decomposition in the special variables $\vartheta = (\omega, Q, \Omega)$. To this end, we write

$$G_J(\eta) = \tilde{G}_J(\vartheta) = g(\omega) + g_1(\omega, Q)Q + O(\Omega). \quad (55)$$

Considering the second relation of (51) in the special variables, we can see that the function $g_1(\omega, Q)$ must be zero. Moreover, the last term $O(\Omega)$ in (55) can be specified to have the form $O(\Omega) = O(\mathcal{P}) + O(\Omega^2)$. We define $G_\Gamma(\eta)$ as

$$G_\Gamma(\eta) = g(\omega). \quad (56)$$

Then,

$$G_J(\eta) = G_\Gamma(\eta) + G_1(\eta), \quad G_1(\eta) = O(\chi) + O(\Phi^2),$$

and, therefore,

$$\begin{aligned} G &= G_\Gamma(\eta) + O(\chi) + O(\Gamma^2), \\ \hat{H}G_\Gamma &= O(\Phi), \quad \{\varphi, G_\Gamma\} = O(\Phi), \quad \{\chi, G_\Gamma\} = O(\chi) + O(\Phi^2), \end{aligned} \quad (57)$$

in virtue of (51).

We now define the variations $\delta_\Gamma\boldsymbol{\eta}$ as

$$\delta_\Gamma\eta = \{\eta, G_\Gamma\}, \quad \delta_\Gamma\lambda_s = \{\bar{\lambda}_s, G_\Gamma\}, \quad \delta_\Gamma\lambda^a = 0. \quad (58)$$

5.3 Constructing the global canonical part of a symmetry

The set $\delta_\Gamma\boldsymbol{\eta}$, G_Γ is an approximate symmetry of the action S_H . Indeed, it obeys the equation

$$\delta_\Gamma\boldsymbol{\eta} \frac{\delta S_H}{\delta \boldsymbol{\eta}} + d_t G_\Gamma = O(\chi) + O(\Gamma^2).$$

However, this approximate symmetry can be modified to become an exact symmetry $\delta_c\boldsymbol{\eta}$, G_c , such that

$$\delta_c\eta = \{\eta, G_c\}, \quad G_c = G_\Gamma + O(\chi) + O(\Phi\Gamma). \quad (59)$$

To justify this assertion, we are going, first of all, to demonstrate that the symmetry equation has a solution with G_c of the form

$$\begin{aligned} G_c &= G_\Gamma + \sum_{i=1}^{N-1} \sum_{s=i+1}^N C_{i|s}^\varphi \varphi^{(i|s)} + \sum_{i=1}^{N-1} \sum_{a=i+1}^N C_{i|a}^\chi \chi^{(i|a)} \equiv G_\Gamma + \sum_{i=1}^{N-1} \mathbf{C}_i \Phi^{(i)}, \\ \mathbf{C}_i &= (C_{i|s}^\varphi, C_{i|a}^\chi) = \mathbf{C}_i(\eta, \lambda^\square). \end{aligned}$$

In the case under consideration, the symmetry equation (44) can be rewritten as

$$\hat{H}G_c - \lambda^a \left\{ \chi^{(1|a|)}, G_c \right\} - \Lambda_s \{ \varphi^{(1|s|)}, G_c \} = \varphi^{(1|s|)} \hat{\delta}_c \Lambda_s + \chi^{(1|a|)} \hat{\delta}_c \lambda^a, \quad \hat{\delta}_c \Lambda_s = \delta_c \lambda_s - \{ \bar{\lambda}_s, G_c \},$$

and then transformed to the form

$$\begin{aligned} \sum_{i=1}^{N-1} \left(C_i \left[\Phi^{(i+1)} + O(\Phi^{(i)}) \right] \right) + \Phi^{(i)} \hat{H} C_i + A &= \varphi^{(1|s|)} \hat{\delta}_c \Lambda_s + \chi^{(1|a|)} \delta_c \lambda^a, \\ A &= \hat{H}G_\Gamma - \lambda^a \left\{ \chi^{(1|a|)}, G_\Gamma \right\} - \Lambda_s \{ \varphi^{(1|s|)}, G_\Gamma \} = O(\chi) + O(\Phi^2), \end{aligned} \quad (60)$$

by using (57). Equation (60) can be analyzed by analogy with equations (33). For example, on the constraint surface $\Phi^{(\dots N-1)} = 0$, we obtain

$$C_{N-1} \Phi^{(N)} + a^{(N)} \Phi^{(N)} = 0, \quad a^{(N)} \Phi^{(N)} = A|_{\Phi^{(\dots N-1)}}.$$

We can choose $C_{N-1} = -a^{(N)}$. In the same manner, we can find all the coefficients C . After substituting these coefficients back into equation (60), we can see that its LHS is proportional to $\Phi^{(1)}$, and, thus, the variations $\delta_c \lambda$ can be found. The coefficients C and the variations $\delta_c \eta$ have the properties

$$C_i|_{G_\Gamma=0} = \delta_c \eta|_{G_\Gamma=0} = 0.$$

In addition, the relation $\hat{\delta}_c \varphi = O(\Gamma) = \{ \varphi, G_c \}$ implies $C^\varphi = O(\Gamma)$. Therefore, the charge G_c has the structure

$$G_c = G_\Gamma + O(\chi) + O(\Phi\Gamma). \quad (61)$$

Note that, in general, the charge G_c depends on λ^{\square} if the Dirac procedure has more than two stages, i.e., $N > 2$; see examples in the Discussion.

5.4 Constructing the gauge and trivial parts of a symmetry

At this stage, we represent the symmetry $\delta\eta$, G as

$$\delta\eta = \delta_c \eta + \delta_r \eta, \quad G = G_c + G_r. \quad (62)$$

Since $\delta_c \eta$, G_c is a symmetry, it is obvious that $\delta_r \eta$, G_r is a symmetry as well. With the help of Eqs. (51) and (61), we can prove the following relations:

$$\begin{aligned} G_r &= \sum_{i=1}^{N_\chi} \sum_{a=i}^{N_\chi} K_{i|a} \chi^{(i|a)} + O(\Gamma^2), \\ \delta_r \eta &= \sum_{i=1}^{N_\chi} \sum_{a=i}^{N_\chi} \left\{ \eta, \chi^{(i|a)} \right\} K_{i|a} + O(\Gamma), \end{aligned} \quad (63)$$

where $K_{i|a} = K_{i|a}(\eta, \lambda^{\square})$ are some LF that do not vanish on the constraint surface.

In turn, we represent the symmetry $\delta_r \eta$, G_r in the form

$$\delta_r \eta = \delta_{\bar{\nu}} \eta + \delta_{\text{tr}} \eta, \quad G_r = G_{\bar{\nu}} + G_{\text{tr}}, \quad (64)$$

where the set $\delta_{\bar{\nu}} \eta$, $G_{\bar{\nu}}$ is a particular gauge transformation with specific gauge parameters,

$$\nu_{(a)} = \bar{\nu}_{(a)} = K_{a|a}, \quad (65)$$

which do not vanish on the constraint surface. This implies

$$G_{\bar{\nu}} = O(\chi) + O(\Gamma^2), \quad \delta_{\bar{\nu}} \eta = \{ \eta, G_{\bar{\nu}} \}. \quad (66)$$

From Eqs. (63) it follows that $\delta_{\text{tr}}\boldsymbol{\eta}$, G_{tr} is a symmetry with the conserved charge of the form

$$G_{\text{tr}} = G'_{\text{tr}} + O(\Gamma^2), \quad G'_{\text{tr}} = \sum_{i=1}^{N_\chi-1} \sum_{a=i+1}^{N_\chi} K_{i|a} \chi^{(i|a)}.$$

We can show that $\delta_{\text{tr}}\boldsymbol{\eta}$, G_{tr} is a trivial symmetry. To this end, we can write for the symmetry $\delta_{\text{tr}}\boldsymbol{\eta}$, G_{tr} the decomposition of the form (45), taking into account that $B_m = O(\Phi)$,

$$\begin{pmatrix} \delta_{\text{tr}}\boldsymbol{\eta}_{\text{tr}} \\ G_{\text{tr}} \end{pmatrix} = \begin{pmatrix} \delta_{\text{tr},J}\boldsymbol{\eta} + O(J) \\ G'_{\text{tr},J} + O(J^2) \end{pmatrix}, \quad G'_{\text{tr},J} = G'_{\text{tr}} + O(\Phi^2). \quad (67)$$

Here, $\delta_{\text{tr},J}\boldsymbol{\eta} = \delta_{\text{tr},J}\boldsymbol{\eta}(\eta, \lambda_\chi^\square)$ and $G'_{\text{tr},J} = G'_{\text{tr},J}(\eta, \lambda_\chi^\square)$. All the relations that take place for the quantities $\delta_J\boldsymbol{\eta}$, G_J hold for the quantities $\delta_{\text{tr},J}\boldsymbol{\eta}$, $G'_{\text{tr},J}$ as well. In particular, the charge G'_{tr} obeys the equation

$$\begin{aligned} \chi^{(1|a)} \delta'_{\text{tr}} \lambda^a &= \hat{H} G'_{\text{tr}} + \lambda^a \left\{ \chi^{(1|a)}, G'_{\text{tr}} \right\} + O(\Phi^2), \\ \delta'_{\text{tr}} \lambda^a &= \delta_{\text{tr}} \lambda^a|_{\Gamma=0} = \delta_{\text{tr},J} \lambda^a + O(\Phi), \end{aligned} \quad (68)$$

which is similar to (49). Equation (68) implies

$$\sum_{i=1}^{N_\chi-1} \sum_{a=i+1}^{N_\chi} \left(\chi^{(i|a)} \hat{H} K_{i|a} + K_{i|a} [\chi^{(i+1|a)} + O(\Phi^{(i)})] \right) = \chi^{(1|s)} \delta'_{\text{tr}} \lambda_s + O(\Phi^2)$$

for the LF $K_{i|a}$, $a = i+1, \dots, N_\chi$. Considering the latter equation on the constraint surface $\Phi^{(N_\chi-1)} = 0$, we obtain

$$K_{N_\chi-1|N_\chi} \chi^{(N_\chi|N_\chi)} = O(\Phi^2) \implies K_{N_\chi-1|N_\chi} = O(\Phi).$$

Substituting the above expression for $K_{N_\chi-1|N_\chi}$ into (68) and considering the resulting equation on the constraint surface $\Phi^{(N_\chi-2)} = 0$, we obtain $K_{N_\chi-2|N_\chi} = O(\Phi)$, and so on. Thus, we can see that $K_{i|a} = O(\Phi)$, $a = i+1, \dots, N_\chi$, and therefore

$$G_{\text{tr}} = O(\Gamma^2). \quad (69)$$

It follows from (68) that

$$\chi^{(1|a)} \delta'_{\text{tr}} \lambda^a = O(\Phi^2) \implies \delta'_{\text{tr}} \lambda^a = O(\Phi) \implies \delta_{\text{tr}} \lambda^a = O(\Gamma). \quad (70)$$

By construction, the transformation $\delta_{\text{tr},J}$ is completely similar to δ_J . Therefore, relation (51) holds for this transformation and implies

$$\delta_{\text{tr},J} \boldsymbol{\eta} = \{\boldsymbol{\eta}, G'_{\text{tr}}\} + O(\Phi) = O(\Phi), \quad \delta_{\text{tr},J} \lambda_s = \{\bar{\lambda}_s, G'_{\text{tr}}\} + O(\Phi) = O(\Phi).$$

Therefore,

$$\delta_{\text{tr}} \boldsymbol{\eta} = O(\Gamma), \quad \delta_{\text{tr}} \lambda_s = O(\Gamma). \quad (71)$$

In the following section, we show that any symmetry of a Hamiltonian action that vanishes on extremals is a trivial symmetry. Therefore, relations (69)–(71) prove that the symmetry $\delta_{\text{tr}}\boldsymbol{\eta}$, G_{tr} is trivial.

We can also prove that the reduction of symmetry variations $\delta\omega$ to the extremals yields global canonical symmetries $\delta_{\text{ph}}\omega$ of the physical action with the conserved charge $g(\omega)$ that is a reduction of the complete conserved charge to the extremals. In addition, any global canonical symmetry $\delta_{\text{ph}}\omega$, $g(\omega)$ of the physical action can be extended to a nontrivial global symmetry $\delta_c\boldsymbol{\eta}$, G_c of the Hamiltonian action S_H . Therefore, classes of nontrivial global symmetries of a singular action are isomorphic to classes of nontrivial global canonical symmetries of the corresponding Hamiltonian action. At the same time, these classes are isomorphic to classes of nontrivial global canonical symmetries of the corresponding physical action.

6 Trivial symmetries

In this section, we are going to prove that any symmetry of the Hamiltonian action that vanishes on the extremals is a trivial symmetry. First, we prove this assertion for a nonsingular Hamiltonian action, and then for the general singular case.

6.1 Nonsingular case

We recall that the Hamiltonian action S_H of any nonsingular theory has the canonical Hamiltonian form

$$S_H[\eta] = \int [p\dot{x} - H(\eta)] dt, \quad \frac{\delta S_H}{\delta \eta} = 0 \implies \dot{\eta} = \{\eta, H\}. \quad (72)$$

Eqs. (72) follow from (14) in the absence of constraints. Below, we are going to prove that:

Symmetries of the canonical Hamiltonian action that vanish on the extremals are trivial symmetries.

To prove this statement, we note that any symmetry $\delta\eta$, G of the Hamiltonian action S_H has to obey the symmetry equation

$$\frac{\delta S_H}{\delta \eta} \delta\eta + d_t G = 0. \quad (73)$$

To analyze the symmetry equation (73), we are going to use, instead of the variables η^\square , the equivalent set of variables η, Γ^\square ; see the comments in the end of sec. 3. Here,

$$\Gamma_A = \frac{\delta S_H}{\delta \eta^A} = E_{AB}^{-1} \dot{\eta}^B - \frac{\partial H}{\partial \eta^A}, \quad E^{AB} = \{\eta^A, \eta^B\}, \quad E_{AB}^{-1} E^{BC} = \delta_A^C. \quad (74)$$

In terms of the new variables, the total time derivative of any LF $F(\eta, \Gamma^\square)$ reads

$$\frac{dF}{dt} = \partial_t F + E^{AB} \left(\Gamma_B + \frac{\partial H}{\partial \eta^B} \right) \frac{\partial F}{\partial \eta^A} + \sum_{k=0} \Gamma_A^{[k+1]} \frac{\partial F}{\partial \Gamma_A^{[k]}}, \quad (75)$$

and the symmetry equation takes the form

$$\{G, H + \epsilon\} + \left(\delta\eta^A + \frac{\partial G}{\partial \eta^B} E^{BA} \right) \Gamma_A + \sum_{k=0} \Gamma_A^{[k+1]} \frac{\partial G}{\partial \Gamma_A^{[k]}} = 0. \quad (76)$$

Let us now suppose that a symmetry $\delta\eta$ vanishes on the extremals, i.e., $\delta\eta = O(\Gamma)$. Representing the charge G as

$$G = G_0 + G_1, \quad G_0 = G|_{\Gamma=0} = G_0(\eta), \quad G_1 = \sum_{m=0} B_m^A(\eta) \Gamma_A^{[m]} + O(\Gamma^2),$$

and considering the terms of zero and first order with respect to the extremals only, we obtain from (76)

$$\{G_0, H + \epsilon\} = 0, \quad \frac{\partial G_0}{\partial \eta^B} E^{BA} \Gamma_A + \sum_{m=0} [\{B_m^A, H + \epsilon\} - B_m^A d_t] \Gamma_A^{[m]} = 0. \quad (77)$$

Analyzing the terms with the extremals Γ^\square (starting from the highest time derivative) in the second equation (77), we can verify that all functions B_m^A are zero. This fact implies that $\partial_t G_0 = 0$. Indeed,

$$B_m^A = 0 \implies \frac{\partial G_0}{\partial \eta^B} = 0 \implies \{G_0, H + \epsilon\} = \{G_0, \epsilon\} = \partial_t G_0 = 0.$$

Taking into account that, in fact, the charge G is defined up to a constant, we can assume $G_0 \equiv 0$. Thus, for any symmetry $\delta\eta$ that vanishes on the extremals, the corresponding conserved charge also vanishes on the extremals and has the form $G = G_1 = O(\Gamma^2)$.

Let us represent this charge as follows:

$$G = \sum_{m=0}^{N<\infty} g_m^A \Gamma_A^{[m]}, \quad (78)$$

where $g_m^A = g_m^A(\eta, \Gamma^\square)$ are some LF. Substituting the representation (78) into (76), we obtain the equation

$$\sum_{m=0}^{N+1} f_m^A \Gamma_A^{[m]} = 0, \quad (79)$$

where

$$f_0^A = \delta\eta^A + d_t g_0^A; \quad f_m^A = g_{m-1}^A + d_t g_m^A, \quad m = 1, \dots, N; \quad f_{N+1}^A = g_N^A. \quad (80)$$

The general solution of equation (79) reads

$$f_k^A = \sum_{m=0}^{N+1} V_{k,m}^{AB} \Gamma_B^{[m]}, \quad (81)$$

where LF $V_{k,m}^{AB} = V_{k,m}^{AB}(\eta, \Gamma^\square)$ are antisymmetric, $V_{m,k}^{BA} = -V_{k,m}^{AB}$. Relation (81) implies for the functions g_k^A the following expressions:

$$g_k^A = - \sum_{s=0}^{N-k} (-d_t)^s \sum_{m=0}^{N+1} V_{k+s+1,m}^{AB} \Gamma_B^{[m]}. \quad (82)$$

Then Eqs. (80)-(82) determine $\delta\eta$ to be

$$\begin{aligned} \delta\eta^A &= \hat{U}^{AB} \Gamma_B = \hat{U}^{AB} \frac{\delta S_H}{\delta \eta^B}, \\ \hat{U}^{AB} &= \sum_{m,k=0}^{N+1} (-d_t)^m V_{m,k}^{AB} (d_t)^k, \quad (\hat{U}^T)^{AB} = -\hat{U}^{AB}, \end{aligned} \quad (83)$$

where \hat{U}^{AB} is an antisymmetric LO. Expression (83) implies that $\delta\eta$ is a trivial symmetry of the action S_H .

6.2 Singular case

We consider here the general case of a singular theory. We are going to prove that the symmetries of the corresponding Hamiltonian action that vanish on the extremals are trivial symmetries.

First of all, we recall that:

a) The Hamiltonian action $S_H[\vartheta]$ of a singular theory (in the special phase-space variables $\vartheta = (\omega, Q, \Omega)$) has the following structure:

$$\begin{aligned} S_H[\vartheta] &= S_{\text{ph}}[\omega] + S_{\text{non-ph}}[\vartheta], \quad \vartheta = (\vartheta, \lambda), \\ S_{\text{ph}}[\omega] &= \int [\omega_p \dot{\omega}_x - H_{\text{ph}}(\omega)] dt, \\ S_{\text{non-ph}}[\vartheta] &= \int [\mathcal{P} \dot{Q} + U_p \dot{U}_x - H_{\text{non-ph}}(\vartheta)] dt, \end{aligned} \quad (84)$$

where

$$\begin{aligned} H_{\text{non-ph}}(\vartheta) &= \lambda_{\mathcal{P}} \mathcal{P}^{(1)} + \lambda_U U^{(1)} + (Q^{(1)} A + Q^{(2...)} B + \omega C) \mathcal{P}^{(2...)} \\ &\quad + \mathcal{P}^{(2...)} D \mathcal{P}^{(2...)} + \mathcal{P}^{(2...)} E U^{(2...)} + U^{(2...)} G U^{(2...)} + O(\vartheta^3). \end{aligned} \quad (85)$$

and A, B, C, E and G are some matrices (see [6]). We recall that the variables ω are canonical pairs, $\omega = (\omega_x, \omega_p)$, of physical variables; the variables Q are nonphysical coordinates; and the variables Ω define the constraint surface by the equations $\Omega = 0$. At the same time, the variables Ω are divided into two groups: $\Omega = (\mathcal{P}, U)$, where U stands for the complete set of SCC (they consist of canonical pairs $U = (U_x, U_p)$) and \mathcal{P} denotes the complete set of FCC. The momenta \mathcal{P} are conjugate to the coordinates Q . The special variables can be chosen so that $\Omega = (\Omega^{(1)}, \Omega^{(2...)})$, where $\Omega^{(1)}$ are primary and $\Omega^{(2...)}$ are secondary constraints. Respectively, $\Omega^{(1)} = (\mathcal{P}^{(1)}, U^{(1)})$, $\Omega^{(2...)} = (\mathcal{P}^{(2...)}, U^{(2...)})$; $\mathcal{P} = (\mathcal{P}^{(1)}, \mathcal{P}^{(2...)})$, $U = (U^{(1)}, U^{(2...)})$; $\mathcal{P}^{(1)}$ are primary FCC; $\mathcal{P}^{(2...)}$ are secondary FCC; $U^{(1)}$ are primary SCC; $U^{(2)}$ are secondary SCC.

b) One can prove that the special phase-space variables can be chosen so that the quadratic part of the non-physical part of the total Hamiltonian takes a simple (canonical) form (see [13]):

$$H_{\text{non-ph}} = \sum_{a=1}^{\aleph_\chi} \left(\sum_{i=1}^{a-1} Q^{(i|a)} \mathcal{P}^{(i+1|a)} + \lambda_P^a \mathcal{P}^{(1|a)} \right) + \lambda_U U^{(1)} + U^{(2...)} F U^{(2...)} + O(\vartheta^3). \quad (86)$$

Here, $(Q, \mathcal{P}) = (Q^{(i|a)}, \mathcal{P}^{(i|a)})$, $\lambda_P = (\lambda_P^a)$, $a = 1, \dots, \aleph_\chi$, $i = 1, \dots, a$, and F is a matrix. We recall that \aleph_χ is the number of stages of the refined Dirac procedure (see [10] and sec. 3) that is necessary to determine all independent FCC from the orthogonal constraint basis. We call such special phase-space variables the superspecial phase-space variables. In the superspecial phase-space variables, the non-physical part of the Hamiltonian action can be written as

$$\begin{aligned} S_{\text{non-ph}} &= S_{\text{non-ph}}^0 + S_{\text{non-ph}}^{\text{int}}, \\ S_{\text{non-ph}}^0 &= \int \left[\mathcal{P} \hat{\Lambda} \mathcal{Q} + \sum_{i=1}^{\aleph_\chi} \mathcal{P}^{(i|i)} \dot{Q}^{(i|i)} + \mathcal{U} \hat{B} \mathcal{U} \right] dt, \quad S_{\text{non-ph}}^{\text{int}} = O(\vartheta^3), \end{aligned} \quad (87)$$

where $\hat{\Lambda}$ and \hat{B} are first-order LO with constant coefficients, and

$$\mathcal{Q} = (\lambda_P^a, Q^{(i|a)}), \quad i = 1, \dots, a-1, \quad a = 1, \dots, \aleph_\chi, \quad \mathcal{U} = (\lambda_U, U).$$

It is important to stress that $[\mathcal{Q}] = [\mathcal{P}]$, due to the fact that $[\lambda_P] = [\mathcal{P}^{(1)}]$. One can see that there exist LO $\hat{\Lambda}^{-1}$ and \hat{B}^{-1} such that

$$\hat{\Lambda} \hat{\Lambda}^{-1} = \hat{\Lambda}^{-1} \hat{\Lambda} = 1, \quad \hat{B} \hat{B}^{-1} = \hat{B}^{-1} \hat{B} = 1.$$

Proceeding to the proof of the statement, we consider the Hamiltonian action \tilde{S}_H (87) in the superspecial phase-space variables. The equations of motion that follow from this action have the form

$$\begin{aligned} \frac{\delta \tilde{S}_H}{\delta \mathcal{U}} &= 0 \implies \mathcal{U} = -\hat{B}^{-1} \frac{\delta S_{\text{non-ph}}^{\text{int}}}{\delta \mathcal{U}}, \quad S_{\text{non-ph}}^{\text{int}} = O(\vartheta^3), \\ \frac{\delta \tilde{S}_H}{\delta \mathcal{Q}} &= 0 \implies \mathcal{P} = -\left(\hat{\Lambda}^T\right)^{-1} \frac{\delta S_{\text{non-ph}}^{\text{int}}}{\delta \mathcal{Q}}, \\ \frac{\delta \tilde{S}_H}{\delta \mathcal{P}} &= 0 \implies \mathcal{Q} = -\hat{\Lambda}^{-1} \left(\frac{\delta S_{\text{non-ph}}^{\text{int}}}{\delta \mathcal{P}} + T \right), \quad T = \left(T^{(i|u)} = \delta_{iu} \dot{Q}^{(u|u)} \right). \end{aligned} \quad (88)$$

These equations allow one to express all the variables \mathcal{U}, \mathcal{P} , and \mathcal{Q} as some LF of ω and $Q^{(i|i)}$, at least perturbatively. Note that, in any case, the exact solution of equations (88) has the form $\mathcal{U} = \mathcal{P} = 0$, $\mathcal{Q} = \psi(\omega, Q^{(i|i)})$, where ψ is an LF of the indicated arguments. Therefore, the variables \mathcal{U} , \mathcal{P} , and \mathcal{Q} are auxiliary ones; see [14, 12]. Excluding these variables from the action \tilde{S}_H , we obtain a dynamically equivalent action $\tilde{S}_H[\omega, Q^{(i|i)}]$. Taking into account the fact that $\mathcal{U} = \mathcal{P} = 0 \implies \Omega = 0$, and relation (23), we obtain

$$\tilde{S}_H[\omega, Q^{(i|i)}] = \tilde{S}_H \Big|_{\mathcal{U}=\mathcal{P}=0, \mathcal{Q}=\psi} = S_{\text{ph}}[\omega]. \quad (89)$$

One ought to keep in mind that equality (89) does not imply the dynamical equivalence of the actions \tilde{S}_H and S_{ph} (and, therefore, equivalence of \tilde{S}_H and S_{ph}); see [9].

Let a transformation $\delta\vartheta, \delta\lambda$, vanishing on the extremals of \tilde{S}_H , be a symmetry of the action \tilde{S}_H . Consider the reduced transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$,

$$(\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}) = (\delta\vartheta, \delta\lambda)|_{\mathcal{U}=\mathcal{P}=0, \mathcal{Q}=\psi}.$$

Obviously, the reduced transformation vanishes on the extremals of the reduced action \tilde{S}_H and is a symmetry transformation of the action \tilde{S}_H . This implies

$$\bar{\delta}\omega = \hat{m} \frac{\delta S_{\text{ph}}}{\delta \omega}, \quad \bar{\delta}Q^{(i|i)} = \left(\hat{n} \frac{\delta S_{\text{ph}}}{\delta \omega} \right)^{(i|i)},$$

where \hat{m} and \hat{n} are some LO. The transformation $\bar{\delta}\omega$ is obviously a symmetry transformation of the nonsingular action S_{ph} that vanishes on its extremals. Therefore, according to the assertion that was proved in the previous subsection, \hat{m} is an antisymmetric LO. Thus, the complete transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$ can be represented in the form

$$\begin{pmatrix} \bar{\delta}\omega \\ \bar{\delta}Q^{(\square)} \end{pmatrix} = \hat{M} \begin{pmatrix} \frac{\delta S_H}{\delta \omega} \\ \frac{\delta S_H}{\delta Q^{(\square)}} \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} \hat{m} & -\hat{n}^T \\ \hat{n} & 0 \end{pmatrix}.$$

Here, obviously, \hat{M} is an antisymmetric matrix.

Finally, the transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$ is a trivial symmetry of the action \bar{S}_H . This implies that the extended transformation $\delta\vartheta$ is a trivial symmetry of the extended action \bar{S}_H , according to the general statement presented in [9].

7 Physical functions in gauge theories

Despite a functional arbitrariness in solutions of the equations of motion for gauge theories, such theories can be used to describe physics. To ensure the independence of physical quantities from the arbitrariness inherent in solutions of a gauge theory, one imposes restrictions on a possible form of physical functions that describe physical quantities.

It was demonstrated (see [6]) that physical LF $A_{\text{ph}}(\boldsymbol{\eta}^\square)$ in the Hamiltonian formulation (we recall that $\boldsymbol{\eta} = (\eta, \lambda), \eta = (q, p)$) have the following structure:

$$A_{\text{ph}}(\boldsymbol{\eta}^\square) = a(\eta) + O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right), \quad (90)$$

where χ stands for the complete set of FCC in the theory, and $a(\eta)$ is a function of the phase-space variables that obeys the following condition:

$$\{a, \chi\} = O(\Phi). \quad (91)$$

We are going to call conditions (90) the physicality conditions in the Hamiltonian sense. It is precisely in this sense that one must understand the usual statement that physical functions must commute with FCC on extremals. In fact, these conditions of physicality are those which are usually called the Dirac conjecture.

On the other hand, it is known that physical functions must be gauge-invariant on the extremals; see, e.g., [6]. Let $\delta_\nu \boldsymbol{\eta}$ be a gauge symmetry of the Hamiltonian action. Then, the gauge variations of the LF $A_{\text{ph}}(\boldsymbol{\eta}^\square)$ must be proportional to the extremals:

$$\delta_\nu A_{\text{ph}}(\boldsymbol{\eta}^\square) = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right). \quad (92)$$

Such a condition will be called the physicality condition in a gauge sense. Until now, it has not been strictly proved whether the two definitions (90), (91) and (92) are equivalent. Using the structure of arbitrary gauge transformation established in Section 4, we are going to prove below an equivalence of these two definitions.

Let an LF $A_{\text{ph}}(\boldsymbol{\eta}^\square)$ be physical in the Hamiltonian sense, i.e., it obeys (90) and (91). Consider its gauge variation $\delta_\nu A_{\text{ph}}$. Such a variation has the form

$$\delta_\nu A_{\text{ph}} = \delta_\nu a(\eta) + O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right). \quad (93)$$

Here, we have used the fact that gauge variations of extremals are proportional to extremals. Taking into account (29) and (39), we can see that the variation $\delta_\nu a_{\text{ph}}$ is proportional to extremals:

$$\delta a = \{a, G_\nu\} = O(\{a, \chi\}) + O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right).$$

Then condition (91) implies (92).

Let an LF $A_{\text{ph}}(\boldsymbol{\eta}^{\square})$ be now physical in the gauge sense, i.e., it obeys (92). We can always represent any LF of $\boldsymbol{\eta}^{\square}$ in terms of a function of the variables η and $\lambda_{\chi}^{\square} = (\lambda^a)^{\square}$ and a function proportional to extremals of the type J (see (24)). Thus, one can always write

$$A_{\text{ph}}(\boldsymbol{\eta}^{\square}) = f(\eta, \lambda_{\chi}^{\square}) + O(J). \quad (94)$$

The condition (92), with allowance made for (29) and (39), implies the equation

$$\{f, G_{\nu}\} + \sum_{m=0}^{m_{\max}} \frac{\partial f}{\partial \lambda^a[m]} \delta_{\nu} \lambda^{a[m]} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right) \quad (95)$$

for the function f . We recall that the variations $\delta_{\nu} \lambda^a$ are given by expression (42). Let us consider such terms in the LHS of (95) that contain (proportional to) the highest time derivatives $\nu_{(a)}^{[a+m_{\max}]}$ of the gauge parameters $\nu_{(a)}$. Since the gauge charge does not contain such derivatives (see (39)), these terms have the form

$$\sum_{a,b=1}^{N_{\chi}} \frac{\partial f}{\partial \lambda^a[m_{\max}]} \mathcal{D}^{ab} \nu_{(b)}^{[b+m_{\max}]} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right), \quad (96)$$

and are proportional to extremals due to (95). In turn, (96) implies the relation

$$\frac{\partial f}{\partial \lambda^a[m_{\max}]} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right)$$

due to the nonsingularity of the matrix \mathcal{D} . Similarly, we can verify that on extremals, the function f does not depend on the variables λ_{χ}^{\square} , i.e.,

$$f(\eta, \lambda_{\chi}^{\square}) = a(\eta) + O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right). \quad (97)$$

Therefore, a physical (in a gauge sense) LF $A_{\text{ph}}(\boldsymbol{\eta}^{\square})$ must have the form (90). Considering equation (92) for such a function, and taking into account (42), we obtain

$$\{a, G_{\nu}\} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right), \quad (98)$$

which implies

$$\left(\sum_{k=1}^{N_{\chi}} \sum_{a=k}^{N_{\chi}} \right) \left(\sum_{m=1}^{N_{\chi}} \sum_{b=m}^{N_{\chi}} \right) \{a, \chi^{(k|a)}\} \mathcal{C}_{ka}^{mb} \nu_{(b)}^{[m-1]} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right). \quad (99)$$

Taking into account the facts that the matrix \mathcal{C} is invertible, the gauge parameters ν are arbitrary functions of time, and thus all ν^{\square} are independent, one can derive from relation (98)

$$\{a, \chi\} = O\left(\frac{\delta S_H}{\delta \boldsymbol{\eta}}\right) \Rightarrow \{a, \chi\} = O(\Phi). \quad (100)$$

Relations (94), (97) and (100) imply that an LF $A_{\text{ph}}(\boldsymbol{\eta}^{\square})$ which is physical in the gauge sense is physical in the Hamiltonian sense as well. Thus, the equivalence of the two definitions of physicality condition is proved.

8 Examples

I. As an example of how gauge symmetries can be recovered from the constraint structure in the Hamiltonian formulation, we consider a field model which includes a set of Yang–Mills vector fields \mathcal{A}_{μ}^a , $a = 1, \dots, r$, and a set of spinor fields $\psi^{\alpha} = (\psi_i^{\alpha}, i = 1, \dots, 4)$,

$$\begin{aligned} S &= \int \mathcal{L} dx, \quad \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + i \bar{\psi}^{\alpha} \gamma^{\mu} \nabla_{\mu\beta}^{\alpha} \psi^{\beta} - V(\psi, \bar{\psi}), \\ G_{\mu\nu}^a &= \partial_{\mu} \mathcal{A}_{\nu}^a - \partial_{\nu} \mathcal{A}_{\mu}^a + f_{bc}^a \mathcal{A}_{\mu}^b \mathcal{A}_{\nu}^c, \quad \nabla_{\mu\beta}^{\alpha} = \partial_{\mu} \delta_{\beta}^{\alpha} - i T_{\alpha\beta}^{\alpha} \mathcal{A}_{\mu}^a, \end{aligned} \quad (101)$$

where V is a local polynomial in the field, which contains no derivatives. The model is based on a certain global Lie group G ,

$$\begin{aligned}\psi(x) &\xrightarrow{g} \exp(i\nu^\alpha T_a)\psi(x), \quad g \in G, \quad \nu^a, \quad a = 1, \dots, r, \\ T_a &= T_a^+, \quad [T_\alpha, T_b] = if_{ab}^c T_c, \quad f_{ab}^k f_{kc}^n + f_{bc}^k f_{ka}^n + f_{ca}^k f_{kb}^n = 0.\end{aligned}$$

For $V = 0$, the action is invariant under the gauge transformations ($\nu^a = \nu^a(x)$)

$$\delta\mathcal{A}_\mu^a = D_{\mu b}^a \nu^b, \quad \delta\psi = iT_a \psi \nu^a, \quad D_{\mu b}^a = \partial_\mu \delta_b^a + f_{cb}^a \mathcal{A}_\mu^c. \quad (102)$$

We assume the polynomial V to be such that the entire action (101) is invariant also under the transformations (102). Below, we relate the symmetry structure of the model to its constraint structure. To this end, we first reveal the constraint structure.

Proceeding to the Hamiltonian formulation, we introduce the momenta

$$p_{0a} = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}^{0a}} = 0, \quad p_{ia} = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}^{ia}} = G_{i0}^a, \quad p_\psi = \frac{\partial_r \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0, \quad p_{\bar{\psi}} = \frac{\partial_r \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0.$$

Hence, there exists a set of primary constraints $\Phi^{(1)} = (\chi_a^{(1)}, \varphi_\sigma^{(1)}, \sigma = 1, 2) = 0$, where

$$\chi_a^{(1)} = p_{0a}, \quad \varphi_1^{(1)} = p_\psi - i\bar{\psi}\gamma^0, \quad \varphi_2^{(1)} = p_{\bar{\psi}}.$$

The total Hamiltonian reads $H^{(1)} = \int \mathcal{H}^{(1)} d\mathbf{x}$,

$$\mathcal{H}^{(1)} = \frac{1}{2}p_{ia}^2 + \frac{1}{4}G_{ik}^{a2} - p_\psi\gamma^0\gamma^k\nabla_k\psi + \mathcal{A}^{0a}(D_{ib}^a p_{ib} - \bar{\psi}\gamma^0 T_a \psi) + V + \lambda_\chi^a \chi_a^{(1)} + \lambda_\varphi^\sigma \varphi_\sigma^{(1)}.$$

By performing the Dirac procedure, one can verify that there appear only secondary constraints $\chi_a^{(2)} = 0$,

$$\begin{aligned}\left\{\varphi_\sigma^{(1)}, H^{(1)}\right\} &= 0 \implies \lambda_\varphi^\sigma = \bar{\lambda}_\varphi^\sigma(\mathcal{A}, \psi, \bar{\psi}), \\ \left\{\chi_a^{(1)}, H^{(1)}\right\} &= 0 \implies \chi_a^{(2)} = D_{ib}^a p_{ib} + i(p_\psi T_a \psi + p_{\bar{\psi}} \bar{T}_a \bar{\psi}), \quad (\bar{T}_a)_\beta^\alpha = -\gamma^0 (T_a^*)_\beta^\alpha \gamma^0.\end{aligned}$$

All constraints φ are of second-class, and all constraints χ are of first-class. It turns out that the complete set of constraints already forms an orthogonal constraint basis, namely,

$$\varphi^{(1|1)} \equiv \varphi^{(1)}, \quad \chi^{(1|2)} \equiv \chi^{(1)}, \quad \chi^{(2|2)} \equiv \chi^{(2)},$$

and there are no constraints $\chi^{(1|1)}$,

$$\begin{array}{ccc}\varphi^{(1|1)} & \rightarrow & \bar{\lambda} \\ \chi^{(1|2)} & \rightarrow & \chi^{(2|2)} \rightarrow O(\Phi)\end{array}.$$

According to the general considerations, we chose the gauge charge in the form

$$G = \int \left[\nu^a \chi_a^{(2|2)} + C^a \chi_a^{(1|2)} \right] d\mathbf{x}, \quad C^a = (c_b^a \nu^b + d_b^a \dot{\nu}^b).$$

Solving the symmetry equation, we obtain $C^a = \dot{\nu}^a - \nu^c \mathcal{A}^{0b} f_{cb}^a = D_{0b}^a \nu^b$. Thus,

$$\begin{aligned}G &= \int \left[p_{\mu a} D_b^{\mu a} \nu^b + i(p_\psi T_a \psi + p_{\bar{\psi}} \bar{T}_a \bar{\psi}) \nu^a \right] d\mathbf{x}, \\ \delta\mathcal{A}_\mu^a &= \{\mathcal{A}_\mu^a, G\} = D_{\mu b}^a \nu^b, \quad \delta\psi = \{\psi, G\} = iT_a \psi \nu^a.\end{aligned}$$

II. Below, we represent a gauge model in which the gauge charge must be constructed with the help of both FCC and SCC.

Consider a Hamiltonian action S_H that depends on phase-space variables q_i, p_i , $i = 1, 2$, and x_a, π_a , $\alpha = 1, 2$, and on two Lagrange multipliers λ_π and λ_p ,

$$S_H = \int \left[p_i \dot{q}_i + \pi_\alpha \dot{x}_\alpha - H^{(1)} \right] dt, \quad H^{(1)} = H_0^{(1)} + V, \\ H_0^{(1)} = \frac{1}{2} \pi_2^2 + x_1 \pi_2 + \frac{1}{2} p_2^2 + \frac{1}{2} q_2^2 + q_1 p_2 + \lambda_\pi \pi_1 + \lambda_p p_1, \quad V = x_1 q_2^2. \quad (103)$$

In what follows, we denote all the variables by $\boldsymbol{\eta} = (x, \pi, q, p, \lambda)$. The model has two primary constraints π_1 and p_1 . One can consider V as the interaction Hamiltonian. It is easy to verify that a complete set of constraints can be chosen as $\chi = (\pi_1, \pi_2)$ and $\varphi = (q_1, q_2, p_1, p_2)$. Here, χ are FCC and φ are SCC. As was already mentioned, gauge symmetries of the action S_H have gauge charges that must be constructed with the help of both FCC and SCC. To demonstrate this fact, let us try to solve the symmetry equation with the gauge charge that is proportional only to FCC. The general form of such a charge can be written as

$$G = A^a \tilde{\chi}_a. \quad (104)$$

Here, $\tilde{\chi}_a$, $a = 1, 2$, are FCC,

$$\tilde{\chi}_a = \pi_a + \psi_a + O(\boldsymbol{\eta}^3), \quad \psi_a = \psi_a(q, p) = O(q^2, qp, p^2), \quad (105)$$

and A^a are some functions of the form

$$A^a = A_0^a + A_1^a(\boldsymbol{\eta}^\parallel) + O(\boldsymbol{\eta}^2), \quad (106)$$

where A_0^a are some functions of time only,⁷ and A_1^a are certain linear LF of the indicated arguments. Thus,

$$G = G_0 + G_1 + O(\boldsymbol{\eta}^3), \quad G_0 = A_0^a \pi_a, \quad G_1 = A_0^a \psi_a + A_1^a \pi_a.$$

The symmetry equation in the case under consideration reads

$$\hat{\partial}_t G + \{G, H^{(1)}\} = O\left(\Phi^{(1)}\right), \quad \hat{\partial}_t = \partial_t + \lambda^{[m+1]} \frac{\partial}{\partial \lambda^{[m]}}. \quad (107)$$

Considering this equation in the zero and first order with respect to the interaction V , we obtain

$$\hat{\partial}_t G_0 + \{G_0, H_0^{(1)}\} = O\left(\Phi^{(1)}\right), \quad (108)$$

$$\hat{\partial}_t G_1 + \{G_0, V\} + \{G_1, H_0^{(1)}\} = O\left(\Phi^{(1)}\right). \quad (109)$$

Equation (108) implies

$$\pi_1 \hat{\partial}_t A_0^1 + \pi_2 \hat{\partial}_t A_0^2 - \pi_2 A_0^1 = O(\pi_1) \implies A_0^1 = \hat{\partial}_t A_0^2. \quad (110)$$

We can see that A_0^2 enter in the solution of the symmetry equation as an arbitrary function of time. In fact, we can identify this function with a time dependent gauge parameter. Taken on the constraint surface $\pi_a = 0$, equation (109) reads

$$\psi_1 \hat{\partial}_t^2 A_0^2 + \left(\hat{\partial}_t \psi_1 + \psi_2 - q_2^2 + \{\psi_1, H_0^{(1)}\} \right) \hat{\partial}_t A_0^2 + \left(\hat{\partial}_t \psi_2 + \{\psi_2, H_0^{(1)}\} \right) A_0^2 = O(p_1). \quad (111)$$

Since the constraints ψ and the Hamiltonian $H_0^{(1)}$ do not depend on the gauge parameter, we obtain from (111) and (105)

$$\psi_1 = O(p_1) = (\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 p_1 + \alpha_4 p_2) p_1,$$

where α are some functions of time. With allowance for this result, we obtain from (111) and (105)

$$\psi_2 = q_2^2 + (\alpha_1 q_1 + \alpha_2 q_2 + \alpha_4 p_2) p_2 + (\beta_1 q_1 + \beta_2 q_2 + \beta_3 p_1 + \beta_4 p_2) p_1, \quad (112)$$

⁷We recall that all LF under consideration may depend on time explicitly; however, we do not include t in the arguments.

where β are some functions of time. Similarly, we have

$$\hat{\partial}_t \psi_2 + \{\psi_2, H_0^{(1)}\} = (2 - \alpha_1)q_1 q_2 + \alpha_1 \lambda_p p_2 + \Delta = O(p_1), \quad (113)$$

where Δ does not contain terms of the form $q_1 q_2$ and $\lambda_p p_2$. Therefore, relation (113) is contradictory.

Thus, we have demonstrated that the symmetry equation has no solutions for the gauge charge of the form (104). The gauge charge must depend on all constraints, both FCC and SCC. An example of such a charge (solution of the symmetry equation (107)) is

$$G = (\pi_2 + q_2^2 + 2q_1 p_2 + 2q_2 p_1 + 2\lambda_p p_1)\nu(t) + (\pi_1 + 2q_1 p_1)\dot{\nu}(t) + O(\eta^3), \quad (114)$$

where $\nu(t)$ is the gauge parameter.

III. As another example, we present below a model with FCC. Here, any nontrivial gauge symmetry of the Hamiltonian action S_H has a gauge charge which depends on Lagrange multipliers. The Hamiltonian S_H and Lagrangian action S of the model have the form

$$\begin{aligned} S_H &= \int dt \left(p_{x^i} \dot{x}^i + p_{y^i} \dot{y}^i + p_{z^i} \dot{z}^i - H^{(1)} \right), \\ S &= \frac{1}{2} \int dt \left[(\dot{y}^i + x^i)^2 + (\dot{z}^i + y^i + g\epsilon_{jk}^i x^j y^k)^2 \right], \end{aligned} \quad (115)$$

where

$$\begin{aligned} H^{(1)} &= H + \lambda^i \Phi_i^{(1)}, \quad H = \frac{1}{2} \left(p_{y^i}^2 + p_{z^i}^2 \right) - x^i p_{y^i} - y^i p_{z^i} - g\epsilon_{jk}^i x^j y^k p_{z^i}, \\ \Phi_i^{(1)} &= p_{x^i}, \quad \epsilon_{jk}^i = -\epsilon_{kj}^i, \quad \epsilon_{12}^i = 1, \quad \forall i; \quad i, j, k = 1, 2, \end{aligned}$$

and g is a constant.

Considering the consistency conditions for the primary constraints $\Phi_i^{(1)}$, we find second-stage constraints $\Phi_i^{(2)}$,

$$\{\Phi_i^{(1)}, H\} = p_{y^i} + g\epsilon_{il}^j y^l p_{z^j} = 0 \implies \Phi_i^{(2)} = p_{y^i} + g\epsilon_{il}^j y^l p_{z^j}.$$

Similarly, we find third-stage constraints $\Phi_i^{(3)}$,

$$\Phi_i^{(3)} = \{\Phi^{(2)}, H\} = \left(\delta_i^j + 2g\epsilon_{li}^j x^l + g\epsilon_{il}^j p_{y^l} \right) p_{z^j}.$$

No more constraints appear from the consistency conditions. All the constraints are FCC. We can replace the constraints $\Phi_i^{(1)}$, $\Phi_i^{(2)}$, and $\Phi_i^{(3)}$ by an equivalent set of FCC, $T_i^{(1)}$, $T_i^{(2)}$, and $T_i^{(3)}$, which is

$$T_i^{(1)} = p_{x^i}, \quad T_i^{(2)} = p_{y^i}, \quad T_i^{(3)} = p_{z^i}.$$

Let us suppose that a $\delta\eta = \{\eta, G\}$ and $\delta\lambda$ is a gauge transformation of the action S_H , with the gauge charge G that does not depend on λ . We can present such a charge as

$$G = \sum_{l=0}^N u^{[l]}(t) G_l(\eta). \quad (116)$$

where $u(t)$ is a function of time. The symmetry equation, in the case under consideration, has the form

$$\partial_t G + \{G, H\} + \{G, p_{x^i}\} \lambda = O(p_x).$$

Since the gauge charge does not depend on λ , for $\lambda = 0$, we obtain from this equation

$$\{G, H\} + \partial_t G = O(p_x). \quad (117)$$

Then

$$\{G, p_{x^i}\} \lambda = O(p_x) \implies \{G, p_{x^i}\} = O(p_x). \quad (118)$$

It follows from (117) that the function $G_N(\eta)$ can be represented in the form

$$G_N = A^i p_{x^i} + O(\Phi^2),$$

where A^i are functions of the coordinates only. Then, considering the terms proportional to $u^{[N]}$ in the symmetry equation, we can see that the function $G_{N-1}(\eta)$ can be presented in the form

$$G_{N-1} = -A^i \Phi_i^{(2)} + B^i p_{x^i} + O(\Phi^2),$$

where B^i are functions of the coordinates only. Relation (118) implies $\partial_{x^j} A^i = 0$. Considering the terms proportional to $u^{[N-1]}$ in the symmetry equation, we can see that the function $G_{N-2}(\eta)$ can be presented in the form

$$G_{N-2} = -D^i \Phi_i^{(2)} + A^i (\delta_i^j + 2g\epsilon_{li}^j x^l) p_{z^j} + C^i p_{x^i} + O(\Phi^2),$$

where

$$D^i \equiv B^i - \partial_t A^i - (x^l \partial_{y^l} + y^l \partial_{z^l} + g\epsilon_{mn}^l x^m y^n \partial_{z^l}) A^i,$$

and C^i are functions of the coordinates only. Considering the terms proportional to $u^{[N-2]}$ in the symmetry equation, we obtain

$$\epsilon_{ij}^l A^i = 0 \implies A^i = 0 \implies D^i = B^i; \quad \partial_{x^j} D^i = 0 \implies \partial_{x^j} B^i = 0.$$

Therefore,

$$G_{N-2} = -B^i \Phi_i^{(2)} + C^i p_{x^i} + O(\Phi^2).$$

Proceeding in the same manner, we obtain, at the final stage of the procedure,

$$G_0 = -M^i \Phi_i^{(2)} + K^i p_{x^i} + O(\Phi^2),$$

where M^i and K^i are functions of the coordinates only. Substituting the functions G_0, \dots, G_{N-1} , and G_N in the above form back to the symmetry equation, we obtain

$$u(t) \left[M^i \Phi_i^{(3)} + (K^i + O(M)) \Phi_i^{(2)} \right] + O(\Phi^2) = O(p_x).$$

Since the constraints $\Phi^{(2)}$ and $\Phi^{(3)}$ are independent, we conclude that $M = K = 0$, which means, in turn, that the charge (116) vanishes quadratically on the extremals, i.e., $G = O(\Phi^2)$. Therefore, the symmetry under consideration is trivial, and a real gauge charge must contain Lagrange multipliers.

IV. Finally, we present below a model with SCC, in which the canonical charge depends on Lagrange multipliers. This model is described by a Hamiltonian action S_H that depends on phase-space variables q_a, p_a , $a = 1, 2$, and x_α, π_α , $\alpha = 1, 2, 3$, as well as on a Lagrange multiplier λ ,

$$\begin{aligned} S_H &= \int \left[p_a \dot{q}_a + \pi_\alpha \dot{x}_\alpha - H^{(1)} \right] dt, \quad H^{(1)} = H_0^{(1)} + V, \quad V = f q_1 x_1 x_3^2, \\ H_0^{(1)} &= \frac{1}{2} (q_a^2 + p_a^2) + x_1 \pi_2 + x_2 \pi_3 + \frac{1}{2} x_3^2 + \frac{1}{2} \pi_2^2 + \frac{1}{2} \pi_3^2 + \lambda \pi_1, \end{aligned} \quad (119)$$

The model has one primary constraint, $\varphi^{(1)} = \pi_1$. One can consider V as the interaction Hamiltonian with a coupling constant f . To demonstrate that the symmetries of the action S_H have charges that must depend on Lagrange multipliers, one has to find all the constraints of the model and then to analyze the symmetry equation. We leave this problem to the reader's consideration.

9 Summary

Below, we summarize the main results of the present article:

A constructive procedure of solving the symmetry equation with the help of a so-called orthogonal constraint basis is proposed.

Using such a procedure, we can determine all the gauge transformations of a given action. In particular, we represent the gauge charge as a decomposition in constraints of the theory; see (26).

Thus, we establish a relation between the constraint structure of the theory and the structure of its gauge transformations. We stress that, in the general case, the gauge charge cannot be constructed with the help of some complete set of FCC alone, since its decomposition contains SCC as well. The gauge charge necessarily contains a part that vanishes linearly in the FCC, and the remaining part of the gauge charge vanishes quadratically on the extremals. With accuracy up to a trivial transformation, any gauge transformation can be represented in the form (29), (30), with the gauge charge (26). The gauge charge contains time derivatives of the gauge parameters whenever there exist secondary FCC. Namely, the order of the highest time derivative that enters the gauge charge is equal to $\aleph_x - 1$, where \aleph_x is the number of the last stage when new FCC still appear.

The above-mentioned procedure of solving the symmetry equation allows one to analyze the structure of any infinitesimal Noether symmetry. Thus, one can see that any infinitesimal Noether symmetry can be represented as a sum of three kinds of symmetries: global, gauge, and trivial symmetries. The global part of a symmetry does not vanish on the extremals, and the corresponding charge does not vanish on the extremals either. The gauge part of a symmetry does not vanish on the extremals, but the gauge charge vanishes on them. The trivial part of any symmetry vanishes on the extremals, and the corresponding charge vanishes quadratically on the extremals. The above division is not unique. In particular, the determination of the global charge from the corresponding symmetry equation, and thus the determination of the global part of a symmetry, is ambiguous. However, the ambiguity in the global part of a symmetry transformation is always the sum of a gauge transformation and a trivial transformation. The reduction of symmetry variations to extremals are global canonical symmetries of the physical action, whose conserved charge is the reduction of the complete conserved charge to the extremals. Any global canonical symmetry of the physical action can be extended to a nontrivial global symmetry of the complete Hamiltonian action.

We note that in our procedure of solving the symmetry equation, the generators of canonical global and gauge symmetries may depend on Lagrange multipliers and their time derivatives. This happens in the case when the number of stages in the Dirac procedure is more than two. We have presented examples of models where generators of canonical and gauge symmetries necessarily depend on Lagrange multipliers.

We have proved that any infinitesimal Noether symmetry that vanishes on the extremals is a trivial symmetry.

Finally, using the revealed structure of arbitrary gauge transformation, we have strictly proved an equivalence of two definitions of physicality condition in gauge theories. One of them states that physical functions are gauge-invariant on the extremals, and the other requires that physical functions commute with FCC (the Dirac conjecture).

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